MATH 8150 (Azoff) Spring 2010

Notes on Symmetry

Definition 1. *Let Γ be a line or circle. We say points $z, z^* \in \mathbb{C}_\infty$ are symmetric with respect to Γ if there is a Mobius transformation T satisfying $T(\Gamma) = \mathbb{R}_\infty$ and $T(z^*) = \overline{T(z)}$.*

Example 2. *Let $a \in \mathbb{C}$. Taking T to be the identity shows that a and $\overline{a}$ are symmetric wrt the real axis.*

Take $Tz = i\frac{z+1}{z-1}$. Substituting and clearing fractions shows that

$$T\left(\frac{1}{\overline{z}}\right) = \overline{Tz}.$$

Since $|z| = 1$ implies $\frac{1}{\overline{z}} = z$, we see that T sends the unit circle to the real axis. Thus the same display shows that $0, \infty$ are symmetric wrt the unit circle as are $a, \frac{1}{\overline{a}}$ for any $a \in \mathbb{C}$. Note the geometric relationship between a and $\frac{1}{\overline{a}}$: they lie on a ray emanating from the origin and the product of their distances from the origin is 1.

Proposition 3. *Suppose z, z^* are symmetric wrt Γ and S is a Mobius transformation. Then $Sz, S(z^*)$ are symmetric wrt $S(\Gamma)$.*

proof. Let T be as in the definition and consider the composite transformation $T \circ S^{-1}$

Example 4. *Taking $Sz = az + b$ we see that z, z^* are symmetric wrt to a line if that line is the perpendicular bisector of the segment joining them. Also ∞ and c are symmetric wrt any circle centered at c.*

The usefulness of symmetry depends on Corolaries 7 and 8 below. To set up these uniqueness results, we need another characterization of those transformations mapping the real axis to itself.

Proposition 5. *Given $x, y \in \mathbb{R}_\infty$ and $v, w \in \mathbb{C}\backslash\mathbb{R}_\infty$, there is a Mobius transformation S which maps $\mathbb{R}_\infty$ into $\mathbb{R}_\infty$ which also satisfies $S(x) = y$ and $S(v) = w$.*

proof. If $x = \infty$, we take T_1 to be the identity; otherwise, we take $T_1(z) = \frac{1}{z-x}$. In either case, we have $T_1(x) = \infty$ and $T_1(v) = \alpha + i\beta$ for some $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$. (My confusion in class resulted from trying to control α and β.) Then we take $T_2(z) = \frac{z}{\beta} - \frac{\alpha}{\beta}$. Then the composite $T_3 := T_2T_1$ has real coefficients and sends v to i and x to ∞. The same construction gives a T_4 with real coefficients which sends w to i and y to ∞. Take $S := (T_4)^{-1}T_3$ to complete the proof.

Corollary 6. *Let $x \in \mathbb{R}_\infty$ and $w \in \mathbb{C}\backslash\mathbb{R}_\infty$. In order that a Mobius transformation map $\mathbb{R}_\infty$ into itself, it is necessary and sufficient that $T(x) \in \mathbb{R}_\infty$ and $T(\overline{w}) = \overline{T(w)}$.*

proof. The condition is necessary since T can be expressed with real coefficients. For sufficiency, apply the preceding proposition to find a Mobius transformation S which maps $\mathbb{R}_\infty$ into itself and also agrees with T on x and w. Because S can be expressed with real coefficients, it also agrees with T on the third point $\overline{w}$ and thus $T = S$.

Corollary 7. *Let w, z, z^* be distinct points in $\mathbb{C}_\infty$. Then there is a unique line or circle Γ passing through w relative to which z, z^* are symmetric.*

proof. For existence, find a Mobius transformation S sending $0, i, -i$ to w, z, z^* respectively. Since $i, -i$ are symmetric wrt to $\mathbb{R}_\infty$, we conclude that z, z^* are symmetric wrt $S(\mathbb{R}_\infty)$.

Suppose now that Γ, Δ are lines or circles passing through w with z, z^* symmetric to both of them. Next write T_Γ, T_Δ for the corresponding Mobius transformations from the Definition. Now apply Proposition 5 to find a Mobius transformation S mapping $\mathbb{R}_\infty$ onto itself which also satisfies $S(T_\Gamma(w)) = T_\Delta(w))$ and $S(T_\Gamma(z)) = T_\Delta(z))$; necessarily we also have $S(T_\Gamma(z^*)) = T_\Delta(z^*))$. Then $S \circ T_\Gamma = T_\Delta$ so

$$\Delta = T_\Delta^{-1}(\mathbb{R}_\infty) = T_\Gamma^{-1}(S^{-1}(\mathbb{R}_\infty)) = T_\Gamma^{-1}(\mathbb{R}_\infty) = \Gamma.$$

Corollary 8. *Let Γ, Δ be lines or circles, and suppose w lies on Γ, while z, z^* are symmetric wrt Γ. If a Mobius transformation T sends w to a point on Δ and z, z^* to points which are symmetric wrt to Δ, then $T(\Gamma) = \Delta$.*

proof. By construction, $T(w)$ lies on $T(\Gamma)$ and $T(z), T(z^*)$ are symmetric wrt $T(\Gamma)$. Thus $T(\Gamma) = \Delta$ by the uniqueness assertion of the preceding corollary.

Notes on Compactness

There is nothing wrong with Conway's treatment of compactness and I should have concentrated on summarizing it. The key result is Theorem 4.9 which provides several equivalent characterizations of the concept.

Theorem 4.9. *The following are equivalent for a metric space* (X, d).

- **a)** *Every open cover of* X *admits a finite subcover (the definition of compactness).*
- **b)** *Every infinite subset of* X *has a limit point.*
- **c)** *Every sequence in* X *has a convergent subsequence.*
- **d)** X *is complete (every Cauchy sequence in* X *converges) + totally bounded (for each* $\epsilon > 0$, X *is covered by finitely many balls of radius* ϵ*).*

Each of these has its advantages and it is important to feel comfortable with them. Conway's proof proceeds a) implies b) implies c) implies d) implies c) implies a). The last implication is deepest, and Conway obtains it as a consequence of Lebesgue's lemma, a result worth understanding in its own right.

The main message I wanted to convey in class is that total boundedness is sometimes easier to work with and almost as good as compactness. These claims are illustrated by Propositions B and A respectively.

Proposition A. *Suppose* X *is a subset of a complete space* Y. *Then* X *is totally bounded iff the closure of* X *in* Y *is compact.*

proof. Note that closures and subsets of totally bounded sets remain totally bounded. Since the closure of X is complete, the result follows from the equivalence of d) and a) of Theorem 4.9.

In fact each metric space can be embedded in a complete metric space, and many authors refer to total boundedness as "relative compactness".

Proposition B involves analogues of a),b),c) of Theorem 4.9 for total boundedness; Lebesgue's Lemma is not needed to establish their equivalence.

The negation of b') provides the most effective way to show a space X is *not* toally bounded – simply find an infinite subset of X so that the set of distances between its distinct points is bounded away from zero. A fortiori, this method also shows X is not compact.

Proposition B. *The following are equivalent for a metric space* X.

- **a)** *Giiven* $\epsilon > 0$, *it is possible to cover* X *by finitely many balls of radius* ϵ *(the definition of totally bounded).*
- **b)** *Given* $\epsilon > 0$ *and an infinite subset* S *of* X, *there is an infinite subset* T *of* S *such that* T *is contained in a ball of radius* ϵ.
- b') *Given* $\epsilon > 0$ *and an infinite subset* S *of* X, *some pair of poinst in* S *must be within* ϵ *of each other.*
- **c)** *Every sequence in* X *has a Cauchy subsequence.*

Proof.

- **a)** $\implies$ b) Assume a) . Fix $\epsilon > 0$ and an infinite subset S of X. In view of a) X can be covered by finitely many balls of radius ϵ, and by the pigeonhole principle at least one of these balls must contain an infinite subset of S.
- **b)** $\implies$ c) Let $f : \mathbb{N} \to X$ be a sequence in X. If some term in this sequence appears infinitely often, then f has a constant subsequence, so we may as well assume f is injective. In view of b), there is an infinite subset S_1 of $\mathbb{N}$ whose image under f is contained in a ball of radius 1. Since $f(S_1)$ is still totally bounded, we can find an infinite subset S_2 of S_1 whose image under f is contained in a ball of radius $\frac{1}{2}$. Continuing inductively, we get a sequence $S_1 \supset S_2 \supset S_3 \ldots$ of infinite subsets of $\mathbb{N}$ with each S_n contained in a ball of radius $\frac{1}{n}$. Now define $g : \mathbb{N} \to \mathbb{N}$ recursively so that g is increasing and $g(n) \in S_n$ for each n. Then $f \circ g$ provides the desired Cauchy subsequence of f.
- **c)** $\implies$ b') Assume b') fails to get an infiniate set S all of whose members are at least ϵ distant from each other. Now let f be an injective function from $\mathbb{N}$ into S. Then f does not admit a cauchy subsequence and c) fails.
- b') $\implies$ a) (This is the simple argument I messed up in class.) Assuming a) fails, we could inductively construct a countably infinite set S all of whose members are at least ϵ distant from one another. But then no ball of radius $\frac{\epsilon}{2}$ could contain more than one member of S, thus showing that b') fails.

Let's close this discussion with a classical result (which will be generalized in Chapter 7 of the text).

Definition. *A collection* $\mathcal{F}$ *of continuous functions mapping a metric space* X *into a metric space* Y *is said to be equicontinuous if given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that we have* $d_Y(f(x), f(y)) < \epsilon$ *whenever* $f \in \mathcal{F}$ *and* $d_X(x, y) < \delta$.

Arzela-Ascoli Theorem. *Let* X *be a compact metric space and equip the space* $C(X)$ *of complex-valued continuous functions with the supremum metric. In order that a subset* $\mathcal{F}$ *of* $C(X)$ *be totally bounded it is necessary and sufficient that*

1. $\mathcal{F}$ *is equicontinuous, and*
2. *For each* $x \in X$, *we have that* $\{f(x) : f \in \mathcal{F}\}$ *is a bounded subset in* $\mathbb{C}$. *(i.e.,* $\mathcal{F}$ *is pointwise bounded).*