MATH 8150 (Azoff) Spring 2010

Notes on Problems from Conway

(IN REVERSE ORDER OF ASSIGNMENTS)

Assignment 12 (Sections 9.1 and 10.1)

9.1.1. Let γ be a non-constant simple closed rectifiable curve with the property that there is a point a such that for all $z \in \{\gamma\}$ the line segment $[a, z] \cap \{\gamma\} = \{z\}$. Define a point w to be **inside** γ if $[a, w] \cap \{\gamma\} = \emptyset$ and let G be the set of all points which are inside γ.

- **a)** Show that G is a region and $\overline{G} = G \cup \{\gamma\}$.
- **b)** Let $f : \overline{G} \to \mathbb{C}$ be continuous and analytic on G. Show that $\int_\gamma f = 0$.
- **c)** Show that $n(\gamma, z) = \pm 1$ if z is inside γ and $n(\gamma, z) = 0$ if $z \notin \overline{G}$.

Proof. The first part of the proof shows that each ray emanating from a intersects the trace of γ exactly once. That makes γ a glorified circle and the rest of the problem follows easily.

The hypothesis tells us no ray emanating from a can intersect $\{\gamma\}$ more than once; in particular a itself cannot lie on γ. To simplify the notation, we take $a = 0$ and assume the domain of γ to be $[0, 2\pi]$. Write $\mathbb{T}$ for the unit circle and define a function $g : \mathbb{T} \to \mathbb{T}$ by $g(\exp(it)) := \frac{\gamma(t)}{|\gamma(t)|}$. The fact that γ is closed tells us that g is well-defined and continuous; the hypothesis tells us that g is injective. By conncectedness, we know the range of g is an arc in $\mathbb{T}$. Choose b in its interior of this arc and express $b = g(c)$. Since $\mathbb{T}\backslash\{c\}$ is connected, the same must be true of its image under g. This forces g to be surjective, i.e. a homeomorphism of $\mathbb{T}$ onto itself. Reparameterizing γ if necessary, we can assume g be the identity map, i.e., $\gamma(t) = r(t)\exp(it)$, where $r(t) = |\gamma(t)|$ is continuous, takes on strictly positive values, and satisfies $r(0) = r(2\pi)$. Geometrically, we think of γ as perturbing the points on the unit circle continuously and radially.

The continuity of r now shows that G and $H := \mathbb{C}\backslash(G \cup \{\gamma\})$ are both open. G is connected because the line segements joining each of its points to the origin lie completely in G. Since $\gamma(t)$ is an endpoint of the line segment joining it to the origin, we see that $G \cup \{\gamma\} \subset \overline{G}$; equality holds because of the openness of H. We have completed the proof of Part a).

For b) it is only necessary to homotop γ radially to the origin. Homotoping γ to a small circle about the origin shows that its winding number about the origin is ± 1 and that $n(\gamma, z) = 0$ for each $z \in H$. Finally an appeal to the continuity of winding number shows that $n(\gamma, z) = \pm 1$ for each $z \in G$ and the proof is ccomplete.

9.1.2. Let G be a region in the plane that does not contain zero and let G^* be the set of all points z such that there is a point $w \in G$ where z and w are symmetric with respect to the circle $|\xi| = 1$.

- **a)** Show that $G^* = \{z : (1/\bar{z}) \in G\}$.
- **b)** If $f : G \to \mathbb{C}$ is analytic, define $f^* : G^* \to \mathbb{C}$ by $f^*(z) = \overline{f(1/\bar{z})}$. Show that f^* is analytic.
- **c)** Suppose that $G = G^*$ and f is an analytic function defined on G such that $f(z)$ is real for $z \in G$ with $|z| = 1$. Show that $f = f^*$.
- **d)** Formulate and prove a version of the Schwarz Reflection Principle where the circle $|\xi| = 1$ replaces $\mathbb{R}$.

Proof. Let R be a Mobius transformation which maps the unit circle onto the extended real axis. We use R and its inverse to translate the problem to the setting of the usual reflection principle. To be concrete, take $Rz = i\left(\frac{1-z}{1+z}\right)$ so that the inverse S of R is given by $Sz = \frac{z-i}{z+i}$.

Recall that two points are symmetric wrt $\mathbb{T}$ iff their images under R are symmetric wrt $\mathbb{R}_\infty$ (and whether this is true does not depend on the choice of R). A routine computation shows that $R(\frac{1}{\bar{z}}) = \overline{Rz}$, thereby showing that $z^* = \frac{1}{\bar{z}}$, and establishing a).

Given an analytic function g we introduce the notation $g^\#$ for the function appearing in the regular reflection principle, i.e., $g^\#(z) := \overline{g(\overline{z})}$. Another brief computation shows that $f^* \circ S = (f \circ S)^\#$, thereby establishing the analyticity of f^* claimed in Part b).

For c), we note that $R(G)$ is symmetric about the real axis and $f \circ S$ is real on the real axis. It follows that $(f \circ S)^\# = f \circ S$ whence $f = f^*$ as desired in c).

Here is the statement for Part d):

Proposition. *Let G be a planar region such that $G = G^*$, that is G is symmetric about the circle $|\xi| = 1$ and assume that G has non-empty intersection with $\mathbb{T}$. Write G_+, G_0, G_- for the parts of G which are respectively inside, on, and outside of $\mathbb{T}$. Suppose $f : G_+ \cup G_0 \to \mathbb{C}$ is continuous on its domain, analytic on G_+ and real valued G_0. Then there is an analytic function $g : G \to \mathbb{C}$ such that $g|_{G_+ \cup G_0} = f$.*

Outline. The fact that $G \subset \mathbb{C}$ means in particular that $0 \notin G$. The argument is carried out by applying the regular reflection principle to the function $f \circ S$ in the domain $R(G)$.

10.1.1. Show that if u is harmonic then so are u_x and u_y.

Comment. The main point to make is that u has derivatives of all orders.

10.1.2. If u is harmonic, then $f = u_x - iu_y$ is analytic.

Comment. Check that u_x and $-u_y$ satisfy the Cauchy-Riemann equations.

Lemma. *Let u be harmonic in a region G and set $V := \{a \in G : u vanishes\ in\ some\ neighborhood\ of\ a\}$. Then either $V = \emptyset$ or $V = G$.*

Proof. It is clear that V is open. In view of connectedness, we need only show that it is closed relative to G. So suppose (z_n) is a sequence in V which converges to a point $b \in G$. Choose $r > 0$ so that $Ball(b, r) \subset G$. Then choose an analytic function f so that $u = \mathcal{R}ef$ in this ball. Finally, find $n \in \mathbb{N}$ so that $z_n \in Ball(b, r)$. Since $z_n \in V$, we conclude u and hence f vanish in a non-empty open subset of $Ball(a, r)$. But that makes f and hence u zero in a neighborhood of b, whence $b \in V$ as desired.

10.1.4. Prove that a nonconstant harmonic function u on a region G is an open map.

Proof. Suppose U is a non-empty open subset of G and fix $a \in U$. Then find $r > 0$ with $Ball(a, r) \subset U$ and express $u = \mathcal{R}ef$ for some analytic function f on this ball. In view of the Lemma, u cannnot be constant on this ball, whence $f(Ball(a, r))$ is open. Since projection on the real axis is a open map from $\mathbb{C}$ to $\mathbb{R}$, we see that $u(Ball(a, r)$ is open as well. Thus $u(U)$, being the union of a family of open sets, must be open as well and the proof is complete.

10.1.5. If f is analytic on G and $f(z) \neq 0$ for any z show that $g = \log |f|$ is harmonic on G.

Solution. Since the composite of a harmonic function with an an analytic function yields another harmonic function, it suffices to show that $(x, y) \mapsto \ln(x^2 + y^2)$ is harmonic on $\mathbb{R}^2\backslash\{0\}$. That can be verified computationally or by appealing to the fact that a branch of the complex logarithm function can be defined in any ball in $\mathbb{C}\backslash\{0\}$.

10.1.6. Let u be harmonic in G and suppose $\overline{B}(a, R) \subset G$. Then

$$u(a) = \frac{1}{\pi R^2} \iint_{\overline{B}(a,R)} u(x, y)\ dx\ dy.$$

Comment. Use polar coordinates.

10.1.9. Let $u : G \to \mathbb{R}$ be harmonic and let $A = \{z \in G : u_x(z) = 0 = u_y(z)\}$; that is A is the set of zeros of the gradient of u. Can A have a limit point in G?

Solution. Trivially yes if G is disconnected or u is constant, but no otherwise. Indeed, suppose G is a region and A has a limit point, choose an open ball about it which is contained in G, and express u as the real part of an analytic function f throughout that ball. Then f' must vanish on G whence u must be constant.

10.1.11. Deduce the Maximum Principle for analytic functions from theorem 1.6.

Solution. One approach is to apply the open mapping theorem to $\ln |f|$.

Alternatively, assume $|f|$ attains a maximum value M at some point $a \in G$. Multiplying f by an appropriate constant of absolute value one,we can assume $f(a) = M$. That, however, implies that the harmonic function $u := \mathcal{R}ef$ attains a maximum value at a. It follows that u and hence f must be constant.

Assignment 11 (Sections 7.1, 7.2, and 7.3)

7.1.1. If (S, d) is a metric space then $\mu(s, t) := \frac{d(s,t)}{1+d(s,t)}$ is also a metric on S. A set is open in (S, d) iff it is open in (S, μ); a sequence is a Cauchy sequence in (S, d) iff it is Cauchy in (S, μ).

Notational Suggestion. Define $f : [0, \infty) \to [0, \infty)$ by $f(t) := \frac{t}{1+t}$. Then $\mu = f \circ d$. Show that f is subadditive and increasing to conclude that μ is a metric. Then get the rest of the problem by noting that f is a homeomorphism of $[0, \infty)$ onto itself.

7.1.4. Let F be a subset of a metric space (X, d) such that $\overline{F}$ is compact. Show that F is totally bounded.

Proof. Let $\epsilon > 0$. According to the definition we must cover F by balls *centered at points of* F.

The collection $\{B(x,\epsilon) : x \in F\}$ covers $\overline{F}$ since $\overline{F} = F \cup \partial F$ and each point in ∂F is arbitrarily close to a point in F. By compactness, there is a finite subcollection $\{B(x_i,\epsilon)\}_{i=1}^n$ which covers $\overline{F} \supset F$ and the proof is complete.

7.1.5. Suppose $\{f_n\}$ is a sequence in $C(G,\Omega)$ which converges to f and $\{z_n\}$ is a sequence in G which converges to a point $z \in G$. Show that $\lim f_n(z_n) = f(z)$.

Proof. For simplicity of notation, assume that $\Omega = \mathbb{C}$ and that $f \equiv 0$. Let $\epsilon > 0$ be given. The most efficient approach (following Ms. Brons) is to set $K := \{z_n : n \in \mathbb{N}\} \cup \{z\}$. Then K is a compact subset of G, whence the sequence (f_n) converges uniformly to 0 on K. Choose $N \in \mathbb{N}$ so that $n \geq N$ implies $||f_n||_K < \epsilon$. In particular, $|f_n(z_n)| < \epsilon$ for $n \geq N$, and the proof is complete.

The triangle inequality, avoided by the assumptions at the beginning of the preceding paragraph, must be made explicit for general Ω.

There are other arrangements of the prrof, but using pointwise convergence is *not* enough. For example, let $f_n : \mathbb{C} \to \mathbb{C}$ by $f_n(z) = \begin{cases} n|z|, & |z| \leq \frac{1}{n}, \\ 2 - n|z|, & \frac{1}{n} \leq |z| \leq \frac{2}{n}, \\ 0, & |z| \geq \frac{2}{n} \end{cases}$ and set $z_n := \frac{1}{n}$. Then (f_n) converges pointwise to the zero function, but $\lim_{n\to\infty} f_n(z_n) = 1$.

7.1.6. Dini's Theorem Consider $C(G,\mathbb{R})$ and suppose that $\{f_n\}$ is a sequence in $C(G,\mathbb{R})$ which is monotonically increasing and $\lim f_n(z) = f(z)$ for all $z \in G$ where $f \in C(G,\mathbb{R})$. Show that $f_n \to f$.

Proof. There is no loss of generality in assuming $f \equiv 0$. Fix $\epsilon > 0$ and a compact subset $K \subset G$. For each $n \in \mathbb{N}$, set $U_n := \{z \in G : f_n(z) > -\epsilon\}$. The hypotheses tell us the $\{U_n\}$ form an open cover of K. Since they are also nested, we can find $N \in \mathbb{N}$ such that $K \subset U_N$. But that means $|f_n(z)| < \epsilon$ for all $n \geq N$ and all $z \in K$ as desired.

7.1.7. Let $\{f_n\} \subset C(G,\Omega)$ and suppose that $\{f_n\}$ is equicontinuous at each point of G. If $f \in C(G,\Omega)$ and $f(z) = \lim f_n(z)$ for each z then show that $f_n \to f$.

Proof. Proposition 7.1.22 tells us the family $\mathcal{F} := \{f_n : n \in \mathbb{N}\}$ is equicontinuous on each compact subset of G and hence normal. Thus every subsequence of (f_n) has a further subsequence which converges in $C(G,\Omega)$ to some function g. Since uniform convergence on compacta implies pointwise convergence, we see that each of these g's is in fact f. We conclude that the full sequence (f_n) converges to f in the metric of $C(G,\Omega)$.

7.1.8a. Let f be analytic on $B(0,R)$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < R$. If $f_n(z) = \sum_{k=0}^{n} a_k z^k$, show that $f_n \to f$ in $C(G,\mathbb{C})$.

Proof. Given a compact set $K \subset B(0,R)$, find $r < R$ with $K \subset \overline{B(0,r)}$. Theroem 3.1.3b tells us that (f_n) converges to f uniformly on $\overline{B(0,r)}$ and hence a fortiori on K.

The proof of Part b) is similar.

7.2.1. Let $f, f_1, f_2, \ldots \in H(G)$ and show that $f_n \to f$ iff for each closed rectifiable curve γ in G, $f_n(z) \to f(z)$ uniformly for $z \in \{\gamma\}$.

Proof. The forward implication is trivial. For the converse, assume $f \equiv 0$ and suppose (f_n) converges uniformly to zero on each circle in G. Given $a \in G$, find $r > 0$ with $B(a,3r) \subset G$. Write $M_n := \sup\{|f_n(w)| : |w-a| = 2r\}$. Then for $|z-a| < r$, Cauchy's estimate yields

$$|f_n(z)| = \frac{1}{2\pi}\left|\int_{|z-a|=2r} \frac{f(w)}{w-z}\,dw\right| \leq \frac{1}{2\pi}\frac{M_n}{r}(2\pi)(2r) = 2M_n.$$

The right hand member of this display is independent of z and approaches 0 as $n \to \infty$. Thus (f_n) converges uniformly to zero in a neighborhood of each point in G. A compactness argument then yields uniform convergence of (f_n) to zero on each compact subset of G whence $f_n \to 0$ as desired.

7.2.6. Show that if $\mathcal{F} \subset H(G)$ is normal then $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is also normal. Is the converse true? Can you add something to the hypothesis that $\mathcal{F}'$ is normal to insure that $\mathcal{F}$ is normal?

Solution. The direct implication follows from Theorem 7.2.1.

The collection of constant functions on G defeats the converse. However if it is assumed that $\mathcal{F}'$ is normal and $\{f(a) : f \in \mathcal{F}\}$ is bounded for a single $a \in G$ then it can be shown $\mathcal{F}$ itself is normal.

7.2.7. Suppose $\mathcal{F}$ is normal in $H(G)$ and Ω is open in $\mathbb{C}$ such that $f(G) \subset \Omega$ for every $f \in \mathcal{F}$. Show that if g is analytic on Ω and is bounded on bounded sets then $\mathcal{G} = \{g \circ f : f \in \mathcal{F}\}$ is normal.

Solution. This is a direct application of Montel's Theorem.

Take $\mathcal{F}$ to be the collection of constant functions taking values $0 < |w| < 1$ and $g(z) = \frac{1}{z}$ to see that the boundedness hypothesis on g is necessary.

7.2.8. Let $\mathbb{D} = \{|z| < 1\}$ and show that $\mathcal{F} \subset H(D)$ is normal iff there is a sequence $\{M_n\}$ of positive constants such that $\limsup \sqrt[n]{M_n} \leq 1$ and if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is in $\mathcal{F}$ then $|a_n| \leq M_n$ for all n.

Proof. Note first that a_n depends on f and is in fact given by $\frac{f^{(n)}(0)}{n!}$. Thus for each n we set $M_n := \sup\{\frac{|f^{(n)}(0)|}{n!} : f \in \mathcal{F}\}$.

Suppose first that $\limsup \sqrt[n]{M_n} \leq 1$ and following Mr. Hicks, set $g(z) := \sum M_n z^n$. Given $r < 1$, the root test tells us that the series $\sum M_n r^n$ converges to some number N_r and thus $|f(z)| \leq N_r$ for each $f \in \mathcal{F}$ and $|z| \leq r$. Thus the family $\mathcal{F}$ is uniformly bounded on compact subsets of $\mathbb{D}$. Montel's Theorem thus tells us that $\mathcal{F}$ is normal as desired.

For the converse, suppose that $\mathcal{F}$ is normal, and fix $r < 1$. Find a number N_r so that $|f(z)| \leq N_r$ for all $f \in \mathcal{F}$ and $|z| \leq r$. Applying Cauchy's esitmate, we thus get for each $f \in \mathcal{F}$, that

$$|f^{(n)}(0)| \leq \frac{n!}{2\pi} \frac{N_r}{r^{n+1}} 2\pi r,$$

whence $M_n \leq \frac{N_r}{r^n}$ for each $n \in \mathbb{N}$. Taking n'th roots thus yields

$$\limsup \sqrt[n]{M_n} \leq \limsup (N_r)^{1/n} \left(\frac{1}{r}\right) = \frac{1}{r}.$$

The arbitrariness of r thus implies $\limsup \sqrt[n]{M_n} \leq 1$ as desired.

7.2.10. Let $\{f_n\} \subset H(G)$ be a sequence of injective functions which converge to f. If G is a region, show that either f is injective or f is constant.

Solution. Suppose f is not constant. Fix $a \in G$ and define a new sequence of functions by $g_n(z) := f_n(z) - f_n(a)$. This sequence converges in $H(G)$ to the non-constant function $g(z) := f(z) - f(a)$. Now note that none of the g_n vanish anywhere in $G' := G \backslash \{a\}$. Apply Hurwitz's Corollary 2.6 to conclude that g is never zero on G' This means that f never takes on the value $f(a)$ in G'. The arbitrariness of a makes f injective.

Assignment 10 (Sections 5.3, 6.1, and 6.2))

5.3.2. Suppose f is analytic on $\overline{B}(0,1)$ and satisfies $|f(z)| < 1$ for $|z| = 1$. Find the number of solutions, counting multiplicities of the equation $f(z) = z^n$ where n is an integer at least 1.

Solution. Write $f(z) = (f(z) - z^n) + (z^n)$ and apply Rouche's Theorem.

5.3.5. Let f be meromorphic on the region G and non constant. Show that neither the poles nor the zeros of f have a limit point in G.

Solution. Note that the set $\{a \in G : f$ vanishes in a neighborhood of $a\}$ is both open and closed relative to G...

5.3.6. Let G be a region and let $H(G)$ denote the set of all analytic functions on G. Show that $H(G)$ is an integral domain. Show that $M(G)$, the meromorphic functions on G is a field.

Solution. The only non-trivial observation is that if $fg \equiv 0$, then the zeros of one of the factors will have a limit point and thus $f \equiv 0$ or $g \equiv 0$.

5.3.7. State and prove a more general version of Rouché's Theorem for curves other than circles in G.

Solution. Apply the *first version* of Cauchy's Theorem.

5.3.9. Let $\lambda > 1$ and show that the equation (*) $\lambda - z - e^{-z} = 0$ has exactly one solution in the half plane $\{z : Re(z) > 0\}$. Show that this solution must be real. What happens to the solution as $\lambda \to 1$?

Solution. Write $\exp(-z) = (\lambda - z - \exp(-z)) + (z - \lambda)$ and apply Rouche's Theorem to see the equation has a unique solution in the half plane. Then apply the intermediate value theorem to see that that solution is in fact real.

To see what happens to the solution as $\lambda \to 1$, define g on $[0, \infty)$ by $g(x) := x + \exp(-x)$. Then $g(0) = 1$ and differentiation shows g is continuous and strictly increasing on its domain. Thus g^{-1} is also a continuous and strictly increasing map from $[1, \infty)$ onto $[0, \infty)$. Thus $\lim_{\lambda \to 1^+} g^{-1}(\lambda) = 0$ and the solutions to Equation (*) approach 0 as $\lambda \to 1$.

5.3.10. Let f be analytic in a neighborhood of $G = \overline{B}(0, 1)$. If $|f(z)| < 1$ for $|z| = 1$, show that there is a unique z with $|z| < 1$ and $f(z) = z$. If $|f(z)| \leq 1$ for $|z| = 1$ what can you say?

Solution. For the first part of the problem, write $f(z) = (f(z) - z) + z$ and apply Rouche's Theorem.

Now suppose only that $|f(z)| \leq 1$ for $|z| = 1$ with equality holding at least once. The Maximum Modulus Principle tells us $|f(z)| \leq 1$ for all $|z| < 1$, and if we ever get equality, then f is constant and thus has no fixed points in the open unit disk $\mathbb{D} := \{z : |z| < 1\}$.

In the remaining case we see that f is a continuous map of the disk to itself and therefore by Schwarz's Lemma, f must reduce to the identity function if it has more than one fixed point. The rotation $f(z) = -z$ shows that f may have a unique fixed point in $\mathbb{D}$; on the other hand, for $0 < |a| < 1$, the Mobius transformation $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$ has no fixed points in $\mathbb{D}$.

6.1.1. Prove the following Minimum Principle. If f is a non-constant analytic function on a bounded open set G and is continuous on $\overline{G}$, then either f has a zero in G or $|f|$ assumes its minimum value on ∂G.

Solution. Assume f never vanishes and consider $\frac{1}{f}$.

6.1.2. Let G be a bounded region and suppose f is continuous on $\overline{G}$ and analytic on G. Show that if there is a constant $c \geq 0$ such that $|f(z)| = c$ for all $z \in \partial G$ then either f is a constant function or f has a zero in G.

Proof. Assume f never vanihes in G. By the maximum and minimum principles, $|f|$ must be constant on G and hence the same is true of f itself.

6.1.4. Let $0 < r < R$ and put $A = \{z : r \leq |z| \leq R\}$. Show that z^{-1} is not the uniform limit of polynomials on A.

Proof. Let γ be a circle in A. Then $\int_\gamma \frac{1}{z}\, dz = 2\pi i$ but every polynomial integrates to 0 over γ.

6.2.1 (Practice). Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic. Then for all $z \in \mathbb{D}$, we have

$$\frac{|f(0)| - |z|}{1 - |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)||z|} \qquad (**).$$

To solve this problem, we need the following somewhat subtle inequality.

Proposition. *Let $a, b \in \mathbb{D}$. Then*

$$\frac{|a| - |b|}{1 - |ab|} \leq \left| \frac{a+b}{1+\bar{a}b} \right| \leq \frac{|a| + |b|}{1 + |ab|}. \qquad (*)$$

Proof. Assume without loss of generality that $|b| \leq |a|$ and that $a \neq 0$. Write θ for the angle between b and a so that $\cos\theta = \frac{\mathcal{R}e\,\bar{a}b}{|ab|}$ and define $g : [0, \pi] \to \mathbb{C}$ by

$$g(\theta) = \left| \frac{a+b}{1+\bar{a}b} \right|^2 = \frac{|a|^2 + |b|^2 + 2|ab|\cos\theta}{1 + |ab|^2 + 2|ab|\cos\theta}.$$

Thus $g'(\theta) = \frac{(1-|a|^2)(1-|b|^2)(-2|ab|\sin\theta)}{|1+\bar{a}b|^4} < 0$ on $(0, \pi)$ whence $g(\pi) \leq g(\theta) \leq g(0)$ for all $\theta \in [0, \pi]$. Taking square roots gives the desired result (*).

Proof of the Problem. Following the hint, let $g : \mathbb{D} \to \mathbb{D}$ by $g(z) := \frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)}$. Then Schwarz' lemma applies to g yielding $|g(z)| \leq |z|$. Substituting for $g(z)$ and applying the left-hand inequality of the Proposition leads to

$$\frac{|f(z)| - |f(0)|}{1 - |f(z)||f(0)|} \leq |z|.$$

Solving for $|f(z)|$ gives the right-hand side of (**).

Similarly, solving $\frac{|f(0)| - |f(z)|}{1 - |f(z)||f(0)|} \leq |z|$ for $|f(z)|$ yields the left-hand side of (**).

Assignment 9 (Section 5.2)

5.2.1a. Evaluate $\int_0^\infty \dfrac{x^2\,dx}{x^4+x^2+1}$.

Shortcut. Identification of poles and evaluation of residues is simplified by noting that $f(z) = \frac{z^2}{z^4+z^2+1} = \frac{z^2(z^2-1)}{z^6-1}$. Writing ω for the principal sixth root of unity, we thus see the poles in the upper half plane are ω and ω^2. Applying l'hopital's rule, we get

$$Res_{z=\omega} f(z) = \lim_{z\to\omega}(z^4-z^2)\frac{z-\omega}{z^6-1} = (\omega^4-\omega^2)\lim_{z\to\omega}\frac{1}{6z^5} = \frac{\omega^5-1}{6}.$$

The residue at $z=\omega^2$ is computed in a similar manner and the answer to the problem is $\frac{\sqrt{3}}{6}$.

5.2.1c. Evaluate $I := \int_0^\pi \frac{\cos 2\theta\ d\theta}{1-2a\cos\theta+a^2}$ where $0 < a^2 < 1$.

Solution. The answer is $I = \frac{a^2}{1-a^2}$.

Hour Exam

4. Let $f : \mathbb{D} \to \mathbb{C}$ be analytic and define $g : (0,1) \to \mathbb{R}$ by

$$g(r) := \sup_{0\le\theta\le 2\pi} |f(r\exp(i\theta))|.$$

Prove that either f is constant or g is strictly increasing.

Proof. Suppose f is not constant and fix $0 < r < s < 1$. Since circles are compact, the maximum value theorem provides us with a complex number w satisfying $|w| = r$ and $|f(w)| = g(r)$. Take h to be the restriction of $|f|$ to the closed disc $|z| \le s$. Since h is not constant, the maximum value and maximum modulus theorem assures us that h attains its maximum value on $|z| = s$ and nowhere else. Thus $g(s) > |f(w)| = g(r)$ as desired.

6. Let (f_n) be a sequence of functions which are analytic on $\mathbb{D}$ and converge uniformly to a function g. Prove that g is analytic on $\mathbb{D}$ and that $\lim_{n\to\infty} f_n'(z) = g'(z)$ for each $z \in \mathbb{D}$.

Second part. Fix $z \in \mathbb{D}$ and choose r with $|z| < r < 1$. Also, for each $n \in \mathbb{N}$, and $|w| = r$, set $F_n(w) : \frac{f_n(w)-g(w)}{(w-z)^2}$. Since $|w-z| \ge r-|z|$ for such w, we conclude that the sequence (F_n) converges uniformly to zero on $|w| = r$. Thus by Cauchy's formula, we get

$$\lim_{n\to\infty} f_n'(z) - g'(z) = \lim_{n\to\infty}\frac{1}{2\pi i}\int_{|w|=r} F_n(w)\,dw = 0,$$

as desired.

Note that the function sequence $f_n(z) = \frac{z^n}{n}$ converges uniformly to zero on $\mathbb{D}$, but the derived sequence does not.

Assignment 7 (Sections 4.5, 4.6, and 4.7)

4.5.1. Suppose $f : G \to \mathbb{C}$ is analytic and define $\phi : G\times G \to \mathbb{C}$ by $\phi(z,w) = \frac{f(z)-f(w)}{z-w}$ if $z \ne w$ and $\phi(z,z) = f'(z)$. Prove that ϕ is continuous and for each fixed w, the map $z \mapsto \phi(z,w)$ is analytic.

Proof. It is not sufficent to show ϕ is continuous in each variable separately.

On the other hand, continuity is clear off the diagonal since division is a continuous function of two variables on $\mathbb{C}\times(\mathbb{C}\backslash\{0\})$.

So fix $a \in \mathbb{C}$ and let $\epsilon > 0$ be given. Apply the continuity of f' to find $\delta > 0$ such that the ball of radius δ about a is contained in G and $|f'(t) - f'(a)| < \epsilon$ whenever $|t-a| < \delta$. Now suppose $||(z,w)-(a,a)|| < \delta$. The Fundamental Theorem of Calculus assures us that

$$\phi(z,w) = \int_0^1 f'(tz+(1-t)w)\,dt$$

when $z \ne w$; note that this remains true when $z = w$ as well. Crashing through with the absolute value, we get

$$|\phi(z,w)-\phi(a,a)| = \left|\int_0^1 f'(tz+(1-t)w) - f'(a)\,dt\right| < \epsilon,$$

and continuity of ϕ is established.

For the second part of the problem, fix $w = b \in G$, and let $g : G \to \mathbb{C}$ by $g(z) := \phi(z, b)$. Expanding f in a Taylor series about b: $f(z) = \sum_{n=0}^{\infty} a_n(z - b)^n$, it follows that

$$g(z) = \sum_{n=1}^{\infty} a_n(z - b)^{n-1}$$

for $z \neq b$. In fact this display is valid for $z = b$ as well since $a_1 = f'(b)$. The proof is completed by recalling that power series define analytic functions.

4.5.4. Show that the integral formula follows from Cauchy's Theorem.

Proof. The proof works for any version of Cauchy's Theorem; we consider the case of a single rectifiable curve.

Let γ be a closed rectifiable curve in a region G with $n(\gamma, w) = 0$ for all $w \in \mathbb{C} \setminus G$ and let $f : G \to \mathbb{C}$ be an analytic function with $a \in G \setminus \{\gamma\}$. Define $\phi : G \to \mathbb{C}$ as in Problem 1. We have shown this to be analytic for fixed w – in particular when $w = a$. Thus,

$$0 = \int_\gamma \phi(z, a)\, dz = \int_\gamma \frac{f(z)}{z - a}\, dz - \int_\gamma \frac{f(a)}{z - a}\, dz = \int_\gamma \frac{f(z)}{z - a}\, dz - 2\pi i f(a) n(\gamma, a).$$

Transposing the last term yields Cauchy's Integral Formula for f itself; Liebnitz's rule gives the formula for higher derivatives.

4.5.5. While this can be done via Cauchy's integral formula, it is also a direct consequence of the fact that $(z - a)^{-n}$has an antiderivative when $n \geq 2$.

4.5.6. Let f be analytic on $D = B(0, 1)$ and suppose $|f(z)| \leq 1$ for $|z| < 1$. Show $|f'(0)| \leq 1$.

Proof. Let $\gamma_r(t) = re^{it}, 0 \leq t \leq 2\pi, 0 < r < 1$. Then, we have

$$|f'(0)| = \left| \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z^2}\, dz \right| \leq \frac{1}{2\pi r^2} 2\pi r = \frac{1}{r}.$$

Letting $r \uparrow 1$ we see that $|f'(0)| \leq 1$ as desired.

4.5.8. Let G be a region and suppose $f_n : G \to \mathbb{C}$ is analytic for each $n \geq 1$. Suppose that (f_n) converges uniformly to a function $f : G \to \mathbb{C}$. Show that f is analytic.

Proof. The converse of Morera's Theorem is not true in general; e.g. $\frac{1}{z}$ is analytic in $\mathbb{C}\backslash\{0\}$ but there are many triangles T with $\int_T \frac{1}{z}\, dz \neq 0$. Fortunately however analyticity is a local property and the converse of Morera's Theorem is valid in open disks. Thus without loss of generality, we assume G is a disk.

As the uniform limit of a sequence of continuous functions, we have that f is at least continuous. Therefore let T be a triangle in G and apply the converse of Morerea's Theorem on the disk G to get

$$\int_T f(z)\ dz = \int_T \lim_{n\to\infty} f_n(z)\ dz = \lim_{n\to\infty} \int_T f_n(z)\ dz = 0$$

Thus f is analytic by Morera's theorem.

4.5.9. Show that if $f : \mathbb{C} \to \mathbb{C}$ is a continuous function which is analytic off $[-1, 1]$ then f is an entire function.

Proof. Let T be a triangle in the plane. If $\{T\} \cap [-1, 1] = \emptyset$ then there is nothing to do as $\int_T f(z)\ dz = 0$. Now, if T does meet $[-1, 1]$ then by suitably decomposing T we reduce to the case where T shares a side or a single point with $[-1, 1]$. Translating T vertically by a small amount ϵ yields a triangle T_ϵ which is disjoint from $[-1, 1]$ and thus $\int_{T_\epsilon} f(z)\, dz = 0$. By continuity of f, we conclude that $\int_T f(z)\, dz = 0$ as well. An appeal to Morera's Theorem completes the proof.

4.6.4. Let $G = \mathbb{C}\backslash\{0\}$ and show that every closed curve in G is homotopic to a closed curve whose trace is contained in $\{z : |z| = 1\}$.

Proof. Let $\gamma_0 : [0, 1] \to G$ be given and define $\gamma_1 : [0, 1] \to G$ by $\gamma_1(t) := \frac{\gamma_0(t)}{|\gamma_0(t)|}$. The desired homotopy can be defined by $\Gamma(s, t) := s\gamma_0(t) + (1 - s)\gamma_1(t)$.

(While other choices of Γ are possible, note that 0 is not allowed to belong to its range. In particular, $n(\gamma_1, 0)$ must equal $n(\gamma_0, 0)$.)

4.6.7 (Practice). Let z_1, z_2, z_3, z_4 be four distinct points lying inside a square and set $f(z) = \frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$. Suppose γ is a counterclockwise parametrization of the boundary of that square. Compute $\int_\gamma f(z)\,dz$.

Solution. The partial fraction decomposition of f takes the form $f(z) = \sum_{j=1}^{4} \frac{a_j}{z-z_j}$. Clearing fractions in this equation, we get $1 = a_1(z-z_2)(z-z_3)(z-z_4) + \dots$ where each of the remaining three terms has a factor of $z - z_1$. Substituting $z = z_1$ into this equation yields $a_1 = \frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}$ with similar expressions for the other a_j. The integral is thus $\sum a_j n(\gamma, z_j) = 2\pi i(a_1 + a_2 + a_3 + a_4)$.

Another approach to the problem would be to decompose the square into four subrectangles, each of which encloses precisely one z_j. Of course, both methods yield the same answer as the residue theorem of Section 5.2, which this problem is intended to motivate.

4.6.10. All integral multiples of π are possible.

4.6.11. The respective answers are $\frac{2\pi i}{3}$, $\frac{4\pi i}{3}$, and $\frac{4\pi i}{3}$.

4.7.1 (Practice). If $f : G \to \mathbb{C}$ is analytic and γ is a rectifiable curve in G, then $f \circ \gamma$ is also rectifiable.

Proof. There is no need to handle the disk case first. Define a function $\phi : G \times G$ as in Problem 4.5.1. The result of that problem tells us that ϕ is continuous on $G \times G$ and hence uniformly continuous on the compact set $K := \{\gamma\} \times \{\gamma\}$. Thus ϕ is bounded on K which makes f Lipschitz on γ. So choose M satisfying $|f(z) - f(w)| \le M|z - w|$ for all $z, w \in \{\gamma\}$.

Now suppose P is a partition of the domain of γ. Then $v(f \circ \gamma; P)| \le Mv(\gamma, P) \le MV(\gamma)$. By definition of least upper bound, we conclude that $V(f \circ \gamma) \le MV(\gamma)$ as desired.

4.7.2 (Practice). Let G be open and suppose that γ is a closed rectifiable curve in G which is homologic to 0 relative to G, i.e., $n(\gamma, a) = 0$ for each $a \notin G$. Set r to be the distance from the trace of γ to (the boundary of) G and $H := \{z \in \mathbb{C} : n(\gamma, z) = 0\}$

1. Show that $\{z : d(z, \partial G) < \frac{r}{2}\} \subset H$.
2. Suppose G is connected and $f : G \to \mathbb{C}$ is analytic, but not constant. Then each equation $f(z) = \alpha$ has at most finitely many solutions z with $n(\gamma, z) \neq 0$.

Proof. (1) Suppose $d(z, \partial G) < \frac{r}{2}$ and find $a \in \partial G$ with $|z - a| = d(z, \partial G)$. By definition of boundary, we can find $b \in G$ with $|b - a| < \frac{r}{2}$ as well. Then the line segment $[z, b]$ is a connected set which doesn't meet $\{\gamma\}$ and thus $n(\gamma, z) = n(\gamma, b)$ which is zero by hypothesis. This puts $z \in H$ as desired.

(2) Find $M > 0$ so that $B(a, M) \supset \{\gamma\}$. Then $n(\gamma, z) = 0$ whenever $|z| > M$. Now consider the set

$$T := \{z \in \mathbb{C} : f(z) = \alpha, |z| \le M, \text{and } d(z, \partial G) \ge \frac{r}{2}\}.$$

This is a compact set, and by part a), must contain every root of $f(z) = \alpha$ having $n(\gamma, z) \neq 0$. The proof is completed by noting that if T were infinite, then it would have a limit point in G, but that would make f constant.

4.7.3. Let f be analytic in $B(a, R)$ and suppose that $f(a) = 0$. Show that a is a zero of multiplicity m iff $f^{(m-1)}(a) = \dots = f(a) = 0$ and $f^{(m)}(a) \neq 0$.

Solution. The easiest approach is to read this off from the Taylor series about a, namely $f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$. Clearly, m is the smallest integer such that $a_m = \frac{f^{(m)}(a)}{m!} \neq 0$.

It is also possible to argue inductively on m.

4.7.4. Suppose that $f : G \to \mathbb{C}$ is analytic and one-to-one. Show that $f'(z) \neq 0$ for all $z \in G$.

Proof. By Corollary 7.6 we see that f^{-1} is analytic. From the chain fule, we therefore get $(f^{-1})'(f(z))f'(z) = 1$ throughout G. This precludes the vanishing of f' at any point of G.

It is also possible to directly apply Theorem 4.7.4 on which the corollary is based.

4.7.7. Use theorem 7.2 to give another proof of the Fundamental Theorem of Algebra.

Proof. Let p be a non-constant polynomial of degree n. Direct computation yields $\lim_{z\to\infty} z\left(\frac{p'(z)}{p(z)} - \frac{n}{z}\right) = 0$. Given $\epsilon > 0$, we can therefore find $M > 0$ so that $\left|\frac{p'(z)}{p(z)} - \frac{n}{z}\right| < \frac{\epsilon}{|z|}$ whenever $|z| \geq M$. Cauchy's extimate thus implies

$$\lim_{R\to\infty}\left[\int_{|z|=R} \frac{p'(z)}{p(z)}\,dz - \int_{|z|=R} \frac{n}{z}\,dz\right] = 0.$$

Since both integrals in this expression are integral multiples of $2\pi i$, Theorem 4.7.2 tells us that for R sufficiently large, the equation $p(z) = 0$ will have precisely n solutions in $|z| < R$.

Assignment 6 (Sections 4.3 and 4.4)

4.3.5. $\cos(a+b) = \cos a\cos b - \sin a\sin b$ fpr every $a, b \in \mathbb{C}$

Proof. You can assume that the identity holds for $a, b \in \mathbb{R}$. Now hold $b \in \mathbb{R}$ fixed and consider the function $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \cos(z+b) - (\cos z\cos b - \sin z\sin b)$. Then f is entire and vanishes on the real axis, a set with a limit point. Thus f vanishes identically. Now fix $a \in \mathbb{C}$ and define $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \cos(a+z) - (\cos a\cos z - \sin a\sin z)$. We have just shown that $g(z) = 0$ for $z \in \mathbb{R}$, and thus g vanishes identically as well, completing the proof.

Alternatively, one can express sin and cos in terms of the exponential function and then apply the homomorphic property of exp.

4.3.10. If f, g are analytic on a region G and $\overline{f}g$ is analytic, then f is constant or g is identically zero.

Proof. Assume g does not vanish identically and choose $a \in G$ with $g(a) \neq 0$. Then $\frac{1}{g}$ exists and is analytic on some open ball U about a. Thus $\overline{f} = (\overline{f}g)(\frac{1}{g})$ is analytic on U along with f itself. But that makes $\mathcal{R}ef$ analytic on U, whence f is constant on U and hence on all of G.

Assignment 5 (Sections 4.1 and 4.2)

4.1.12. Let $\gamma : [0,\pi] \to \mathbb{C}$ by $\gamma(t) = r\exp(it)$ and define $I(r) := \int_\gamma \frac{\exp(iz)}{z}\,dz$. Then $\lim_{r\to\infty} I(r) = 0$.

Proof. Fix $\epsilon > 0$. We have

$$|I(r)| \leq \int_0^\pi \exp(-r\sin t)\,dt = \int_0^\epsilon \exp(-r\sin t)\,dt + \int_\epsilon^{\pi-\epsilon} \exp(-r\sin t)\,dt + \int_{\pi-\epsilon}^\pi \exp(-r\sin t)\,dt.$$

The first and last integrals are less than ϵ and the middle integrand is bounded above by $\exp(-r\sin\epsilon)$. Thus $|I(r)| \leq 2\epsilon + \pi(\exp(-r\sin\epsilon))$. For each fixed ϵ, we have $\lim_{r\to\infty}\exp(-r\sin\epsilon) = 0$, whence $\limsup_{r\to\infty}|I(r)| \leq 2\epsilon$. The arbitrariness of ϵ thus gives the desired result.

4.2.2. Leibniz's Rule for complex functions.

Theorem. *Let γ be a rectifiable curve in a region G. Suppose that $\phi : \{\gamma\} \times G \to \mathbb{C}$ is continuous and define $g : G \to \mathbb{C}$ by $g(z) := \int_\gamma \phi(w,z)\,dw$.*

1. *g is continuous on G.*
2. *If the partial derivative ϕ_2 exists and is continuous on $\{\gamma\} \times G$ then g is analytic on G and $g'(z) = \int_\gamma \phi_2(w,z)\,dw$ for each $z \in G$.*

Proof. Fix $a \in G$ and choose $r > 0$ so that the closed ball B of radius r about a is contained in G. Then $\{\gamma\} \times B$ is compact so ϕ is uniformly continuous on it. Let $\epsilon > 0$ be given and choose $0 < \delta < r$ so that $|\phi(w,z) - \phi(w,a)| < \epsilon$ whenever $|z - a| < \delta$.

Now suppose $|z - a| < \delta$. Then

$$|g(z) - g(a)| = \left|\int_\gamma \phi(w,z) - \phi(w,a)\,dw\right| < \epsilon V(\gamma),$$

and the first part of the Theorem is established.

Assume now that ϕ_2 exists and is continuous on $\{\gamma\} \times G$ and define $\psi : \{\gamma\} \times B \to \mathbb{C}$ by

$$\psi(w,z) = \begin{cases} \frac{\phi(w,z)-\phi(w,a)}{z-a}, & z \neq a \\ \phi_2(w,a), & z = a \end{cases}.$$

By the Fundamental Theorem of Calculus, we have $\psi(w,z) = \int_0^1 \phi_2(w, tz+(1-t)a)\,dt$ when $z \neq a$; note that this formula is also valid when $z = a$. Thus the continuity of ϕ_2 yields continuity of ψ.

Now define $h : B \to \mathbb{C}$ by $h(z) := \int_\gamma \psi(w,z)\,dw$. The first part of the problem tells us that h is continuous at a and thus

$$\lim_{z\to a} \frac{g(z)-g(a)}{z-a} = \lim_{z\to a} h(z) = h(a) = \int_\gamma \phi_2(w,a)\,dw.$$

and we have established differentiability of g at a. Finally, continuity of g' follows from the last display by continuity of ϕ_2.

4.2.3. Suppose γ is rectifiable and h is continuous on $\{\gamma\}$. Define a function g on $G := \mathbb{C}\backslash\{\gamma\}$ by $g(z) := \int_\gamma \frac{h(w)}{w-z}\,dw$. Then g is analytic on its domain and successive derivatives of g can be found by differentiating inside the integral.

Proof. Define $\phi : \{\gamma\} \times G \to \mathbb{C}$ by $\phi(w,z) := \frac{h(w)}{w-z}$. Then ϕ satisfies the hypotheses of the preceding problem and thus the current result follows from the result of that problem.

4.2.8. Use the fact that Cauchy's Theorem holds in disks to show that it also holds in half-planes.

Solution. This problem is less important now that we know more general versions of Cauchy's Theorem.

A quick and sneaky approach would be to show that every curve contained in an open half-plane is actually contained in an open disk contained in that half plane.

But the author's instructions were to use a Mobius transformation. So suppose $\gamma : [0,1] \to H$ is a closed piecewise-smooth curve in a half-plane H and f is analytic in H. Fix a Mobius transformation T which maps the open unit disk $\mathbb{D}$ onto H.

Define $F : \mathbb{D} \to \mathbb{C}$ by $F(z) = f(T(z))T'(z)$ and $\Gamma : [0,1] \to \mathbb{D}$ by $\Gamma(t) = T^{-1}(\gamma(t))$. Then F is analytic on $\mathbb{D}$ and Γ is a piecewise smooth path in $\mathbb{D}$. Since $T \circ \Gamma = \gamma$, the chain rule yields

$$0 = \int_\Gamma F(z)\,dz = \int_0^1 F(\Gamma t)\Gamma'(t)\,dt = \int_0^1 f(\gamma(t))T'(\Gamma(t))\Gamma'(t)\,dt = \int_0^1 f(\gamma(t))\gamma'(t)\,dt = \int_\gamma f(z)\,dz.$$

This establishes Cauchy's Theorem for piecewise-smooth curves in H. The proof is completed by an appeal to Lemma 4.1.19.

4.2.11b (Practice). Set $f(z) = \dfrac{1}{2i}\log\left(\dfrac{1+iz}{1-iz}\right)$. Show that f is a branch of the arctangent function.

Proof. Fix z in the domain of f and write $w = f(z)$. Then $\exp(2iw) = \frac{1+iz}{1-iz}$ and

$$\tan(f(z)) = \tan w = \frac{\exp(iw)-\exp(-iw)}{2i(\exp(iw)+\exp(-iw))} = \frac{\exp(2iw)-1}{i(\exp(2iw)+1)}.$$

Substituting for $\exp(2iw)$ in this display yields $\tan(f(z)) = z$ as desired.

Assignment 4 (Section 3.3)

3.3.9. Find the most general Mobius transformation that maps the unit circle Γ onto itself.

Solution. The easist approach is via symmetry. There must be some a that T sends to 0. The absolute value of a cannot be 1. We assume for definiteness that $a \neq \infty$; the remaining case is easy. The point symmetric to a wrt the unit cirlcle is $\frac{1}{\bar{a}}$ and T must send this to ∞ because that is the point symmetric to 0 wrt the unit circle. Thus

$$Tz = c\frac{z-a}{z-\frac{1}{\bar{a}}} = d\frac{z-a}{1-\bar{a}z},$$

where c, d are constants. Now $1 = |T(1)| = |d|$ and we have found a necessary condition for T to map Γ to itself. Noting that a complex number has absolute value one iff its conjugate coincides with its reciprocal, we see that this condition is also sufficient.

3.3.10 (Practice). Find all Mobius transformations that map the unit disc onto itself.

Solution. Since such transformations must map the unit circle onto itself, it is only necessary to determine which transformations of the preceding problem satisfy $|T(0)| < 1$. The answer is $Tz = d\frac{z-a}{1-\bar{a}z}$ where $|a| < 1$ and $|d| = 1$.

3.3.11 (Practice). Suppose Γ is a line or circle while (z_2, z_3, z_4) and (w_2, w_3, w_4) are each triples of distinct points lying on Γ. Write T, S for the unique Mobius transformations sending these points to $1, 0, \infty$ respectively. Finally suppose z, z^* have the property that $T(z^*)$ is the complex conjugate of Tz. Then $S(z^*)$ is the complex conjugate of Sz as well.

Proof. The map $S \circ T^{-1}$ sends $\mathbb{R}_\infty$ to itself. Thus it can be written with real coefficients and hence respects complex conjugation. Thus

$$S(z^*) = (ST^{-1})T(z^*) = (ST^{-1})(\overline{Tz}) = \overline{ST^{-1}T(z)} = \overline{Sz},$$

as desired.

(Note: In the cross product notation, (z_1, z_2, z_3, z_4) stands for the image of z_1 under the unique Mobius transformation which sends z_2, z_3, z_4 to $1, 0, \infty$ respectively. This explains the text's restatement of the problem.)

3.3.16. Map $G := \mathbb{C}\backslash[-1, 1]$ onto the open unit disk via an analytic function f. Can f be one-to-one?

Solution. Start with the Mobius transformation $Tz := \frac{z-1}{z+1}$. This maps the segment $[-1, 1]$ to the ray $[-\infty, 0]$ and sends ∞ to 1, so G goes to the complementary domain in $\mathbb{C}$. Follow with the principal branch of the square root function and the Mobius transformation S which sends 1 to $\frac{1}{2}$, -1 to 2 and 0 to 1. By symmetry, S sends the imaginary axis onto the unit circle. Write g for the composite of what we have so far. Then g imaps G bijectively onto $\mathbb{A} := \mathbb{D}\backslash\{\frac{1}{2}\}$. The squaring map sends $\mathbb{A}$ onto $\mathbb{D}$ so the construction is completed by taking $f(z) = (g(z))^2$.

Since g is invertible, the question at the end of the problem is equivalent to asking whether there is an injective analytic map h of $\mathbb{A}$ onto $\mathbb{D}$. The answer is no. Here is an argument using results from later chapters. Assume such an h existed. Since h is bounded, it must have a removable singularity at $\frac{1}{2}$ and thus can be extended to an analytic function H defined on all of $\mathbb{D}$. Since H is not constant, the open mapping theorem would then force $H(\frac{1}{2}) \in \mathbb{D}$. Thus $H(\frac{1}{2}) = h(a)$ for some $a \in \mathbb{A}$. But then a second application of the open mapping theorem would contradict the injectivity of h.

3.3.17. If an analytic function f maps a region G into a circle, then f must be constant.

Solution. By connectedness, it is enough to check that f is locally constant, so we may as well assume that $f(G)$ is properly contained in the circle. Find a mobius transformation T which maps the circle into $\mathbb{R}_\infty$ with a point outside the range of f going to ∞. Thus $T \circ f$ maps G into $\mathbb{R}$. That forces $T \circ f$ to be constant and hence f is constant as well.

Note: The result of this problem and many others like it are immediate consequences of the open mapping theorem of Section 4.7.

Assignment 3 (Sections 3.1 and 3.2)

3.1.3. Let (a_n) and (b_n) be sequences of real numbers. Assuming the right hand side of the display makes sense, we have

$$\limsup(a_n + b_n) \le \limsup a_n + \limsup b_n. \qquad (*)$$

Proof. For each $n \in \mathbb{N}$, write $A_n := \sup_{k \ge n} a_k$, $B_n := \sup_{k \ge n} b_k$, and $C_n := \sup_{k \ge n} a_k + b_k$. Fix $k, n \in \mathbb{N}$ with $k \ge n$. By definition of upper bound, we have $a_k + b_k \le A_n + B_n$. Thus we have $C_n \le A_n + B_n$ by definition of *least* upper bound. Also the sequences $(A_n), (B_n), (C_n)$ are non-increasing. Thus they each converge in $\mathbb{R} \cup \{\pm\infty\}$

The right hand side of the display does not make sense when $\lim A_n = \infty$ and $\lim B_n = -\infty$ or vice versa. Otherwise if some $A_n = \infty$, then $A_n = \infty$ for all n and (*) is immediate; the argument is the same if some $B_n = \infty$. In the remaining case, both eequences (a_n) and (b_n) are bounded above and we have $\lim C_n \le \lim(A_n + B_n) = \lim A_n + \lim B_n$ as desired.

3.2.13 (Bonus). Bernoulli Numbers (This problem is nteresting but not all that important.)

a) Find the power series expansion about zero of the function $g(z) = \frac{\exp z - 1}{z}$ and determine its radius of convergence.

Solution. The power series expansion for the exponential function is $\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Subtracting 1, dividing by z and shifting the index of summation, we get $g(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$. The radius of convergence of this series is infinite.

b) Take $f(z) := \frac{1}{g(z)}$ with Maclaurin series $f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!}$. Find its radius r of convergence.

Solution. This is the distance from 0 to the smallest solution to $g(z) = 0$, i.e., the smallest non-zero solution to $\exp z = 1$. Thus $r = 2\pi$.

c) Show that

$$0 = a_0 + \binom{n+1}{1} a_1 + \cdots + \binom{n+1}{n} a_n \qquad (*)$$

for each $n \in \mathbb{N}$.

Solution. Use the identity $f(z)g(z) = 1$ and equate coefficients of like powers of z.

d) Show that $h(z) = f(z) + \frac{z}{2}$ is an even function and conclude that $a_k = 0$ for odd $k > 1$.

Solution. $h(-z) = \frac{-z}{\exp(-z)-1} - \frac{z}{2} = h(z)$. It follows that the even index coefficients in the Maclaurin series of h must vanish.

e) The numbers $B_{2n} := (-1)^{n+1} a_{2n}$ are called the *Bernoulli numbers.* Compute $B_2, \ldots, B_{10}$.

Solution. Use Equation (*) recursively.

3.2.18. Suppose $f, g : G \to \mathbb{C}$ are branches of z^a and z^b respectively. Then $f \cdot g$ is a branch of z^{a+b}. If f, g map G into itself, then $f \circ g$ is a branch of z^{ab}.

Proof. Some restriction needs to be made. For example, take $G := \{z \in \mathbb{C} : \mathcal{R}e(z) > 0\}$ with $f(z), g(z)$ to be the branches of the square root function with positive and negative real parts respectively. Then $f(z)g(z) = -z$ for all $z \in G$, and this is not a branch of z^1. One way out is to assume that f, g use a common branch L of the logarithm function. Then $\exp((a+b)L(z)) = \exp(aL(z))\exp(b(L(z)) = f(z)g(z)$ as desired.

This fix is consitent with the third paragraph from the bottom of Page 40 where the author suggests that his power functions will always be defined in terms of the prinicpal branch log of the logarithm function. This convention, along with the hypothesis that f, g map G into itself, also yield the second part of the problem. Indeed $\{\mathcal{I}m(b \log z) : z \in G\}$ is a connected subset of $\mathbb{R}$ which is not allowed to include an odd multiple of π. Thus there is a fixed integer k such that $b \log z - 2\pi i k$ belongs to the range of log for all $z \in G$. This implies $\log(\exp(b \log z)) = b \log z$ for all $z \in G$ and allows us to simplify $f(g(z)) = \exp(a \log(\exp(b \log z)))$ to $\exp(ab \log z)$.

Assignment 2 (Chapter 2)

2.1.11 (Bonus). In order that $S := \{\mathrm{cis}(k\theta) : k \in \mathbb{N}\}$ be dense in the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, it is necessary and sufficient that θ be an irrational multiple of π.

Proof. If θ is a rational multiiple of π, then S is finite, and necessity is established.

For the converse, suppose that $\frac{\theta}{\pi}$ is irrational. Then S must be infinite and thus has a limit point in $\mathbb{T}$. Since S is a semigroup, this means 1 is a limit point of $S \cup S^{-1}$. Replacing S by S^{-1} if necessary, we may assume that 1 is a limit point of S itself. It is then easy to see that S is dense in $\mathbb{T}$.

2.2.5 (Bonus). Let (X, d) be a metric space and $\epsilon > 0$. Say that $a, b \in X$, are *ϵ-incremental* if there are points $z_0, z_1, \ldots, z_n \in X$ with $z_0 = a$, $z_n = b$, and $d(z_{j-1}, d_j) < \epsilon$ for $1 \le j \le n$.

a) If X is connected, then every pair of points in X is ϵ-incremental for every $\epsilon > 0$.
b) The converse of Part a) can fail even if X is complete.
c) The converse of Part a) holds when X is compact.

The introduction of a subset F of X in the statement of the problem is distracting since connectedness of F does not depend on whether we consider it a subset of itself or as a subset of the ambient space X. The reference to "closure" is also irrelevant because F is always a closed subset of itself.

(I made up Part c) and the terminology "ϵ-incremental".)

Proof.

a) Suppose X is connected, and fix $\epsilon > 0$ and $a \in X$. Then set $S := \{b \in X :$ the pair a, b is ϵ- incremental$\}$. Check that S is non-empty, open and closed. By connectedness, $S = X$, completing the proof.

b) Take A to be the x-axis in $\mathbb{R}^2$ and $B := \{(x, y) \in \mathbb{R}^2 : x > 0 \;\&\; xy = 1\}$. Then $X : A \cup B$ is a closed subset of $\mathbb{R}^2$ and hence is complete. It is also easy to check that each pair in X is ϵ-incremental for each $\epsilon > 0$. However A, B are non-empty disjoint closed subsets of X so X is not connected.

c) Suppose X is compact and every pair in X is ϵ-incremental for each $\epsilon > 0$. Now suppose $X = A \cup B$ with A, B non-empty and closed. The hypothesis implies that for each $n \in \mathbb{N}$, we can find $a_n \in A$ and $b_n \in B$, with $d(a_n, b_n) < \frac{1}{n}$. Dropping down to a subsequence if necessary, we can assume that the sequence (a_n) converges to some point $c \in A$. But then (b_n) converges to c as well. By closure, $c \in A \cap B$. Thus A, B cannot be disjoint and we have shown that X is connected.

2.3.5 (not assigned). Every convergent sequence is Cauchy.

Proof. Suppose the sequence (a_n) converges to L in a metric space (X, d) and let $\epsilon > 0$ be given. Find $N \in \mathbb{N}$ such that $n \ge N$ implies $|d(a_n, L) < \frac{\epsilon}{2}$. Now suppose $m, n \ge N$. Then

$$|a_m - a_n| = |(a_m - L) + (L - a_n)| \le |a_m - L| + |a_n - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Assignment 1 (Chapter 1)

1.3.3. Let $a \in \mathbb{R}$ and $c > 0$ be fixed. Describe the set of points z satisfying the equation

$$|z - a| - |z + a| = 2c \qquad (*).$$

Solution. The triangle inequality tells us that $|z - a| \leq |z + a| + 2|a|$. Thus the set of z satisfying Equation (*) will be empty when $c > |a|$.

If $c = |a|$, then (*) requires equality in the triangle inequality, and $z + a$ must be a non-negative multiple of $-2a$, so the solution set for z is a ray emanating from $-a$ on the real axis.

Finally, if $c < |a|$, then the graph is one branch of the hyperbola $\frac{x^2}{c^2} - \frac{y^2}{a^2 - c^2} = 1$ where as usual, $z = x + iy$. This can be derived by algebraic manipulation of the equation (*) or recalling the definition of a hyperbola in terms of distances from a pair of foci.