Review

4.3 #16

$\dim V^{\perp} = n - \dim V$
$4 - 2 = 2$

let $V = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^4$ (v_1, v_2)

set $T = \text{Proj}_V$

find standard matrix of T

Plan: (for this section)

1) find convenient basis $\mathcal{B}$ for $\mathbb{R}^4$
2) find $[T]_{\mathcal{B}}$
3) use change of basis formula

1) pick $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

$T\vec{v}_1 = \vec{v}_1$ $\quad T\vec{v}_2 = \vec{v}_2$

b/c they are in V we know what proj. onto V does to them

$\vec{v}_3 =$ $\quad \vec{v}_4 =$ } orthogonal to V

to find $\vec{v}_3$, $\vec{x}_4$ consider $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \perp V$

$\Longleftrightarrow \quad x_1 + 2x_3 + x_4 = 0$

$x_2 - x_3 + x_4 = 0$

$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$ $\quad x_1 = -2x_3 - x_4$
$x_2 = x_3 - x_4$
$x_3 = x_3$
$x_4 = x_4$

orthogonal compliment is span of v_3 & v_4

so $\vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ $\quad \vec{v}_4 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

$T\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad T\vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

do T to 1st basis vector
express T_{v_1} as linear combination of other vectors

$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 1 & -1 \\ 2 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$P[T]_{\mathcal{B}} = [T]_{STD} P$$

$$[T]_{STD} = P[T]_{\mathcal{B}} P^{-1}$$

$\boxed{\text{To find } [Proj_V]_{STD}}$ $\rightsquigarrow$ many diff ways

Construct a matrix A whose column space is V

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$[proj_{C(A)}]_{STD} = A(A^TA)^{-1}A^T$$

where comes from

Solve equation $A\vec{x} = b$ which usually does not have a solution so...
consider $A^TA\vec{x} = A^T\vec{b}$ which always has a solution
$\iff \quad A\vec{x} = proj_{C(A)}\vec{b}$

Find an orthogonal basis for V

take $\vec{w}_1 = \vec{v}_1$

part of v_2 orthogonal to v_1

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{-1}{6}\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/6 \\ 0 \\ 1/3 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1 \\ -2/3 \\ 7/6 \end{bmatrix}$$

would be nice to avoid fractions,
so can use a scalar multiple of w_2 $\to$ replace with

$$6\begin{bmatrix} 1/6 \\ 1 \\ -2/3 \\ 7/6 \end{bmatrix}$$

new basis for V : $\begin{bmatrix}1\\0\\2\\1\end{bmatrix}, \begin{bmatrix}1\\6\\-4\\7\end{bmatrix}$ } orthogonal basis for V

find $\text{proj}_V \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \text{proj}_{\begin{bmatrix}1\\0\\2\\1\end{bmatrix}} \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} + \text{proj}_{\begin{bmatrix}1\\6\\-4\\7\end{bmatrix}} \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$

$$= \frac{x_1 + 2x_3 + x_4}{6}\begin{bmatrix}1\\0\\2\\1\end{bmatrix} + \frac{x_1 + 6x_2 - 4x_3 + 7x_4}{102}\begin{bmatrix}1\\6\\-4\\7\end{bmatrix}$$

$$[\text{Proj}_V]_{STD} = \left[\text{Proj}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \dots, \text{Proj}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]$$

$$\text{Proj}_V \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \frac{1}{6}\begin{bmatrix}1\\0\\2\\1\end{bmatrix} + \frac{1}{102}\begin{bmatrix}1\\6\\-4\\7\end{bmatrix}$$

$V = \text{span}\left\{\begin{bmatrix}1\\0\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\2\end{bmatrix}\right\}$

b/c this is 3 dim
easy to use $V^\perp$ b/c 1 dim

take $T = \text{proj}_V$
find $[T]_{STD}$

① Find a basis for $V^\perp$ $\longrightarrow$ let vectors be rows of A then

$$\begin{bmatrix}1&0&2&1\\0&1&-1&1\\0&0&1&2\end{bmatrix} \rightarrow \begin{bmatrix}1&0&0&-3\\0&1&0&3\\0&0&1&2\end{bmatrix}$$

$x_1 = 3x_4$
$x_2 = -3x_4$
$x_3 = -2x_4$
$x_4 = x_4$

basis for $V^\perp$ $\begin{bmatrix}3\\-3\\-2\\1\end{bmatrix}$

$$\text{Proj}_{V^\perp}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \frac{\begin{bmatrix}3\\-3\\-2\\1\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}}{\left\|\begin{bmatrix}3\\-3\\-2\\1\end{bmatrix}\right\|^2}\begin{bmatrix}3\\-3\\-2\\1\end{bmatrix} = \frac{3x_1 - 3x_2 - 2x_3 + x_4}{23}\begin{bmatrix}3\\-3\\-2\\1\end{bmatrix}$$

$$\text{proj}_V \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} - \frac{3x_1 - 3x_2 - 2x_3 + x_4}{23} \begin{bmatrix} 3 \\ -3 \\ -2 \\ 1 \end{bmatrix}$$

$$[\text{Proj}_{V^\perp}]_{STD} = \begin{bmatrix} 9/23 & \\ -9/23 & \\ -6/23 & \\ 3/23 & \end{bmatrix}$$

see what transformation $(\text{proj}_{V^\perp})$ does to: $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

$$[\text{Proj}_V]_{STD} = I - [\text{Proj}_{V^\perp}]_{STD}$$

b/c $\text{Proj}_V + \text{Proj}_{V^\perp} = I$

Given:

$\mathcal{B}$: $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\2\end{bmatrix}$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

find the standard matrix of T

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[T]_{STD}\, P = P\, [T]_{\mathcal{B}}$$

$$[T]_{STD} = P [T]_{\mathcal{B}} P^{-1}$$

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what does it mean $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

by defn : $T\vec{v}_1 = \vec{v}_1 + \vec{v}_2$ $\qquad T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$

$T\vec{v}_2 = \vec{v}_1$ $\qquad T\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

set $A = [T]_{STD}$

then $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$ and $A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

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$$A \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

Find the parabola $y = ax^2$ (*) which comes closest to passing through the points $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

really want: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to statisfy (*)

$$1 = a$$

also want $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ to statisfy (*)

$$3 = 4a$$

$$\underbrace{\begin{bmatrix} 1 \\ 4 \end{bmatrix}}_{A} \underset{\vec{x}}{[a]} = \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\vec{b}} \quad \Big\} \text{ inconsistent}$$

for the least squares solution we solve

$$A^T A \vec{x} = A^T \vec{b}$$

$$[1 \;\; 4] \begin{bmatrix} 1 \\ 4 \end{bmatrix} x = [1 \;\; 4] \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$17x = 13$$

$$x = 13/17$$

$$\hookrightarrow a = 13/17$$

$\therefore$ best parabola is $y = 13/17\, x^2$

In text

(*) $\text{reflec}_{v} = \text{proj}_{v} - \text{proj}_{v^\perp}$

$= \text{proj}_{v} - (I - \text{proj}_{v})$

$= 2\,\text{proj}_{v} - I$

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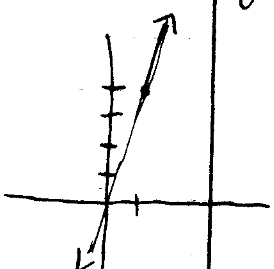
let T = reflection about line $y = 4x$

find $[T]_{STD}$



① Find the formula for $(\text{proj}_v)\begin{bmatrix} x \\ y \end{bmatrix} = \text{proj}_{\begin{bmatrix} 1 \\ 4 \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix} =$

$\frac{x+4y}{17}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

what happens to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$[\text{Proj}_v]_{STD} = \begin{bmatrix} 1/17 & 4/17 \\ 4/17 & 16/17 \end{bmatrix}$

if you square matrix should get same matrix b/c projecting 2x gives same vector

by (*)

$[\text{reflc}_v]_{STD} = \begin{bmatrix} 2/17 & 8/17 \\ 8/17 & 32/17 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$=$

when you reflect 2x you get back were u started from so matrix $^2 = I$

other plan:

choose convenient basis:

we know $T\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$,

$T\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

orthogonal to $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$P[T]_{\mathcal{B}} = [T]_{STD}\,P$

$[T]_{STD} = P\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}P^{-1}$

pick basis $\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ $\quad [T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}$

$\vec{x}_1, \vec{x}_2, \dots \vec{x}_K$ are orthogonal if $\vec{x}_i \cdot \vec{x}_j = 0$ for $i \neq j$.

$\vec{x}_1, \vec{x}_2, \dots, \vec{x}_K$ are orthonormal if in addition each vector is a unit vector (has length $(\|x\|) = 1$)

Good things about orthogonal bases

① It's easy to find coordinates relative to them

Ex.

$\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix}$ <- orthogonal basis for $\mathbb{R}^2$

express $\begin{bmatrix}5\\17\end{bmatrix}$ as a linear combination of these vectors.

which means we want : $\begin{bmatrix}5\\17\end{bmatrix} = c_1 \begin{bmatrix}1\\2\end{bmatrix} + c_2 \begin{bmatrix}2\\-1\end{bmatrix}$

dot with $\begin{bmatrix}1\\2\end{bmatrix}$

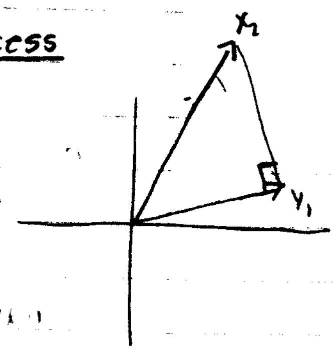
$$\begin{bmatrix}1\\2\end{bmatrix} \cdot \begin{bmatrix}5\\17\end{bmatrix} = c_1 \begin{bmatrix}1\\2\end{bmatrix} \cdot \begin{bmatrix}1\\2\end{bmatrix} + \underbrace{c_2 \begin{bmatrix}2\\-1\end{bmatrix} \cdot \begin{bmatrix}1\\2\end{bmatrix}}_{\text{zero}}$$

$$39 = 5c_1$$
$$c_1 = 39/5$$

now dot with $\begin{bmatrix}2\\-1\end{bmatrix}$

$$-7 = 5c_2$$
$$c_2 = -7/5$$

How do you find them? → Gram-Schmidt Process



start with $\vec{x}_1, \vec{x}_2, \vec{x}_3$ independent

find a new set of vectors.

orthogonal
same span
as x_1, x_2, x_3

$$\vec{y}_1 = \vec{x}_1$$

$$\vec{y}_2 = \vec{x}_2 - \underbrace{\frac{\vec{x}_2 \cdot \vec{y}_1}{\|\vec{y}_1\|^2} \vec{y}_1}_{\text{proj onto } y_1}$$

so you are left w/ proj orthogonal to y_1

$$\vec{y}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{y}_1}{\|\vec{y}_1\|^2} \vec{y}_1 - \frac{\vec{x}_3 \cdot \vec{y}_2}{\|\vec{y}_2\|^2} \vec{y}_2$$

Ⓐ Show $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$ is similar to a *diagonal matrix*.

has basis of eigenvectors

Ⓑ find A^{100}

A ① find the eigenvalues

$$\det(A - tI) = \det \begin{bmatrix} -t & 0 & 0 \\ 0 & 2-t & 2 \\ 0 & 2 & 2-t \end{bmatrix} =$$

$$(-t) \det \begin{bmatrix} 2-t & 2 \\ 2 & 2-t \end{bmatrix} =$$

$$(-t)\left[4 - 4t + t^2 - 4\right] =$$

$$(-t)(t^2 - 4t) =$$

$$(-t)(t)(t-4) = 0$$

$$\left.t = 0,\ t = 4\right\} \text{ eigenvalues}$$

find the eigenvectors: (*basis for each eigenspace*)

for $t = 4$

$$A - 4(I) = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$
$x_2 = x_3$
$x_3 = x_3$

basis for E_4: $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

one free variable hence 1 basis vector

for $t = 0$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_1$
$x_2 = -x_3$
$x_3 = x_3$

2 free variable hence 2 basis vectors

basis for E_0: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

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$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$AP = PD$$

$$A = P \begin{bmatrix} 0 & & \\ & 0 & \\ & & 4 \end{bmatrix} P^{-1}$$

$$A^{100} = P \begin{bmatrix} 0 & & \\ & 0 & \\ & & 4^{100} \end{bmatrix} P^{-1}$$

} can leave like this unless you need it for something

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don't worry about kernel & image

kernel ↓ $N(A)$, image ↓ $C(A)$

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suppose $A^2 = I$

$$\det A^2 = \det I$$

$$(\det A)^2 = 1$$

$$x^2 = 1$$

$$x = \pm 1$$

$A^T A = I$ (orthogonal matrix)

$$\det(A^T A) = \det I$$

$$(\det A^T)(\det A) = 1$$

$$(\det A)(\det A) = 1$$

$$(\det A)^2 = 1$$

$$\det A = \pm 1$$

→ matrices of rotations & reflections are orthogonal matrices

they preserve length

Prop: TFAE:

(1) $A^T A = I$

(2) $\|Ax\| = \|x\|$ for every vector x

PF:

$$\|Ax\|^2 = (Ax)\cdot(Ax) = (Ax)^T(Ax) = x^T A^T A x = x^T x = x \cdot x = \|x\|^2$$

proofs w/ eigenvalues & eigenvectors
w/ similarity
w/ subspaces & independence

similar matrices

all proj onto 1 dim space r similar to one another

$A, B \in M_n$ are similar if *there is an invertible* matrix P such that $P^{-1}AP = B$ $(AP = PB)$

properties:

(1) $[T]_\beta$ is *similar to* $[T]_{STD}$

(2) a matrix A is *diagonizable IFF* A *is similar to* a diagonal matrix IFF $\exists$ basis *of eigenvectors for* A

remember:

A similar to $B \overset{?}{\Longrightarrow} A^2$ *similar to* B^2

Proof: *find invertible* P *with*

$$AP = PB$$

$$A^2P = A(AP) = A(PB) = (AP)(B) = P(BB) = PB^2$$

or

$$P^{-1}AP = B$$

$$(P^{-1}AP)^2 = B^2$$

$$(P^{-1}AP)(P^{-1}AP) = B^2$$

$$P^{-1}A^2P = B^2$$

$\therefore$ statement true

A^2 similar to $B^2 \overset{?}{\Longrightarrow}$ A similar to B

NO : $A = [1]$ $B = [-1]$
not similar, but A^2 even $= B^2$

if something is similar to I then has to be I

To show A, B are similar:

1) find a linear transformation T with $[T]_{STD} = A$ and $[T]_{\mathcal{B}} = B$

2) solve $AP = PB$ for invertible P

To show A, B are NOT similar:

1) show that they have different eigenvalues
2) show the eigenspaces have different dimensions

will not have to do

Ex.

all continuous functions

① $V = \{ f \in C(\mathbb{R}) \mid f(3) = 1 \}$

not a subspace because $\vec{0}$ is not includc

② $V = \{ f \in C(\mathbb{R}) \mid f(3) = 0 \}$

Yes this is a subspace

(1) $0 \in V$ → so when you apply zero function to 3 do you get zero → yes!

(2) suppose $f, g \in V$
then $(f+g)(3) = f(3) + g(3) = 0 + 0 = 0$
$\therefore$ V is closed under addition

(3) suppose $f \in V$ and $c \in \mathbb{R}$
$(cf)(3) = c(f(3)) = c(0) = 0$
$\therefore cf \in V \therefore$ V is closed under scalar multiplication

③ $V = \{ f \in C(\mathbb{R}) \mid f(3) \geq 0 \}$

1) $0 \in V$
2) closed under addition
3) $f(x) = x$ (belongs to V)
$-1(f(x)) = -x$
$(-f)(3) = -3$

not a subspace b/c not closed under scalar multiplication polynomials of degree ≤ 2

(4) $V = \{ p \in P_2 \mid p(3) = 0 \}$

$P_2 = \{\text{all polynomials of deg} \leq 2\}$

basis for P_2 : $P_1(x) = 1$
$P_2(x) = x$
$P_3(x) = x^2$

clearly they span

suppose $c_1(1) + c_2 x + c_3 x^2 = 0$ for all x

in particular, true $x = 0$

then $c_1 = 0$

so $c_2 x + c_3 x^2 = 0 \quad \forall x$

$x = -1$

$-c_2 + c_3 = 0$

$x = 1$

$c_2 + c_3 = 0$

force $c_2 = c_3 = 0$

since $c_1 = c_2 = c_3 = 0$, vectors are independent

$\dim P_2 = 3$

b/c # vectors needed to make basis for vector space = dim

what is the most general member of P^2?

$P(x) = ax^2 + bx + c$

when is $P \in V$?

$\Longleftrightarrow P(3) = 0$

$\Longleftrightarrow 9a + 3b + c = 0$

$\Longleftrightarrow c = -9a - 3b$

most general member of V : $p(x) = ax^2 + bx - 9a - 3b$

$= a(x^2 - 9) + b(x - 3)$

basis for V:

$$P_1(x) = x^2 - 9$$
$$P_2(x) = x - 3$$

clearly they span

✓ if independent

suppose $c_1(x^2-9) + c_2(x-3) \equiv 0$ for $\underline{\underline{\text{all } x}}$

take $x = -3$

$c_2(-6) = 0 \implies c_2 = 0$

now have $c_1(x^2-9) = 0$

take $x = 0$

$c_1(-9) = 0 \implies c_1 = 0$

or

$$c_1 \underline{\underline{x^2}} + c_2 \underline{\underline{x}} + (-9c_1 - 3c_2)\underline{\underline{(1)}} = 0$$

we know these are indep

$\therefore$ indep $\therefore$ basis

$\dim V = 2$

40,000 girls in a class

$3/4$ of the girls who color their hair one day also color it the next day

$1/5$ of the girls who go natural one day color their hair the next day

everyone goes natural the zeroth day

1) How many girls have colored their hair on day n?
2) find the equilibrium # of
3) do you always approach this equilibrium?

x_n = # with colored hair on day n

y_n = # with natural hair on day n

(1

$$x_{n+1} = 3/4\, x_n + 1/5\, y_n$$
$$y_{n+1} = 1/4\, x_n + 4/5\, y_n$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 3/4 & 1/5 \\ 1/4 & 4/5 \end{bmatrix}}_{A} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

1) <u>diagonalize A</u>

$$\det(A - tI) = \left(\left(\tfrac{3}{4} - t\right)\left(\tfrac{4}{5} - t\right)\right) - \tfrac{1}{20}$$
$$= t^2 - \tfrac{31}{20} t + \tfrac{11}{20}$$
$$= (t-1)(t + {}^{11}/_{20})$$

eigenvalue	eigenvector
1	$\begin{bmatrix} 4 \\ 5 \end{bmatrix}$
$11/20$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

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$$A - {}^{11}/_{20} I = \begin{bmatrix} 1/5 & 1/5 \\ 1/4 & 1/4 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 11/20 \end{bmatrix}$$

$$P = \begin{bmatrix} 4 & 1 \\ 5 & -1 \end{bmatrix}$$

$$AP = PD$$

$$A = P\begin{bmatrix} 1 & 0 \\ 0 & 11/20 \end{bmatrix} P^{-1}$$

on day n, $\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} 0 \\ 40{,}000 \end{bmatrix} = P\begin{bmatrix} 1 & 0 \\ 0 & 11/20 \end{bmatrix}^n P^{-1} \begin{bmatrix} 0 \\ 40{,}000 \end{bmatrix}$

2) equilib. $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ must be an eigenvector corresponding to eigenvalue $\lambda = 1$

$\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ must be some $c\begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$$x_\infty + y_\infty = 40{,}000$$

$$4c + 5c = 40{,}000 \implies 9c = 40{,}000$$

$$c = \frac{40{,}000}{9}$$

$4c = \frac{160{,}000}{9} \approx 20{,}000$ have hair dyed

3) $\lim_{n\to\infty} \begin{bmatrix} x_n \\ y_n \end{bmatrix} = P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \underbrace{\begin{bmatrix} 4/9 & 4/9 \\ 5/9 & 5/9 \end{bmatrix}}_{\text{Azoff did for us}} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} =$

$\begin{bmatrix} 4/9\,(x_0 + y_0) \\ 5/9\,(x_0 + y_0) \end{bmatrix} = \begin{bmatrix} 4/9\,(40{,}000) \\ 5/9\,(40{,}000) \end{bmatrix}$ ← same # as equilib #

$x_0 + y_0 = 40{,}000$

so no matter starting numbers, would have same equilibrium.

continuous
1st derivative cont.
2nd derivative cont.

$$V = \{ f \in C^2(\mathbb{R}) \mid f'' - f' - 6f = 0 \}$$

Is V a subspace? YES

1) $0 \in V$
2) suppose $f, g \in V$
consider $(f+g)'' - (f+g)' - 6(f+g) =$
$f'' + g'' - f' - g' - 6f - 6g =$
$(f'' - f' - 6f) + (g'' - g' - 6g) =$
$0 + 0 = 0 \quad \therefore f+g \in V$

$\therefore$ V is closed under addition

3) suppose $f \in V$ and $c \in \mathbb{R}$
consider $(cf)'' - (cf)' - 6(cf) =$
$c(f'' - f' - 6f) = 0$

Fact: V is 2 dimensional

Find a basis: wouldn't be that hard

try $f(x) = e^{rx}$
want $f'' - f' - 6f = 0$

$$r^2 e^{rx} - r e^{rx} - 6 e^{rx} = (r^2 - r - 6)\underbrace{e^{rx}}_{\text{never zero}}$$

so $r^2 - r - 6 = 0$
$(r-3)(r+2) = 0$
$r = 3, \; r = -2$

$f_1(x) = e^{3x}$
$f_2(x) = e^{-2x}$

✓ if they are independent

if 2 thing are always = then their derivatives are equal

suppose $c_1 e^{3x} + c_2 e^{-2x} \equiv 0$
$x = 0$

$c_1 + c_2 = 0$ $\qquad c_1 3e^{3x} + c_2(-2)(e^{-2x}) = 0$ if $x = 0$

$3c_1 - 2c_2 = 0 \Rightarrow c_1 = c_2 = 0$