MATH 2250 (Azoff) Fall 2009

Solutions to Paper Supplement on Section 3.1

(See below for solutions to practice test.)

1 Let $f(x) = 4x + 2x^2$. (Functions vary.)

a) *Find constants A, B, C satisfying $\frac{f(x+h)-f(x)}{h} = Ah + Bx + C$.*

No limit is involved in this part – only substitution in the function f and algebraic simplification. However I did not take off if you combined both parts of the problem as long as you used correct notation.

$$\frac{f(x+h)-f(x)}{h} = \frac{4(x+h)+2(x+h)^2-4x-2x^2}{h} = \frac{4h+4xh+h^2}{h} = 4+4x+h,$$

as long as $h \neq 0$. Matching this up with the desired form, we see $A = 1$ and $B = C = 4$.

b) *Use your answer to Part a) to find $f'(x)$.*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 4+4x+h = 4+4x.$$

2 Let $f(x) = 4x^2 - 5x + 12$. (Functions vary.)

a) *According to the definition of derivative, $f'(x) = \lim_{t\to x} \dots$.*

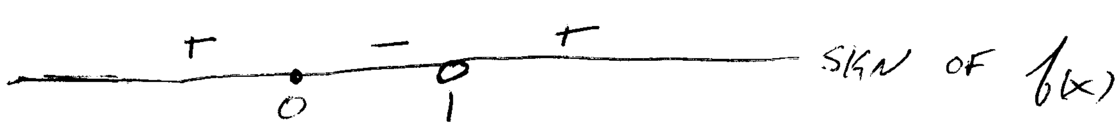
The dots should be replaced by $\frac{f(t)-f(x)}{t-x} = \frac{4t^2-5t+12-4x^2+5x-12}{t-x}$ and you can enter the latter expression in the box. To simplify this expression in preparation for Part b), rearrange the numerator to get $\frac{4(t^2-x^2)-5(t-x)}{t-x} = 4(t+x)-5$. You can also use either of the latter expressions as your answer to Part a).

b) *Use your answer to Part a) to find $f'(x)$.*

$$f'(x) = \lim_{t\to x} 4(t+x) - 5 = 8x - 5.$$

3 *Apply the intermediate value theorem to solve the inequality $\frac{x^3-2x^2+x}{x-1} \geq 0$.* Set $f(x) := \frac{x^3-2x^2+x}{x-1}$. The only point of discontinuity of this function is $x = 1$ and its only root is $x = 0$. These are the only places where $f(x)$ could change sign. Since $f(2) = 2 > 0$, we see that $f(x) > 0$ for $x > 1$, from $f(\frac{1}{2}) = \frac{-1}{4}$, we conclude that $f(x) < 0$ for $0 < x < 1$, and from $f(-1) = 2 > 0$, we get $f(x) > 0$ for $x < 0$.

This gives us the following sign chart.



Thus the answer to the question is $(-\infty, 0] \cup (1, \infty)$.

4 A ball is thrown up from the top of a cliff. Its height (in feet) above the canyon below t seconds later is given by the formula $y = f(t) = 432 + 96t - 16t^2$.

a) *What is the maximum height above the canyon floor achieved by the ball ?*

Using either the definition or short method, we see that the velocity of the ball is given by $f'(t) = 96 - 32$. At maximum height, the velocity of the ball is zero; this happens when $t = 3$ seconds. Thus the maximum height is $f(3) = 576$ feet.

b) *When does the ball hit the floor of the canyon ?*

This happens when $f(t) = 0$. There are two solutions to this equation, but the given formula is only valid physically at one of them, namely $t = 9$ seconds.

c) *What is the speed of the ball just before it hits the floor of the canyon ?*

(Thanks to Mr. Berl for pointing out the limit.) The velocity is given by $\lim_{t \to 9^-} 96 - 32t = -192$ ft/sec. Thus the speed is $|-192| = 192$ ft/sec.

Solutions to Practice Test

1 (20 points) Compute the following limits. If a limit does not exist, explain why.

a) $\lim_{x \to 5} \frac{x^2 - 25}{3x + 1} = \frac{0}{16} = 0$. (We can just plug in $x = 5$ because the function is continuous there.)

b) $\lim_{x \to 3^-} \frac{|x - 3|}{x^2 - 9} = \lim_{x \to 3^-} \frac{-(x - 3)}{x^2 - 9} = \lim_{x \to 3} \frac{-1}{x + 3} = \frac{-1}{6}$. (When $x < 3$, we have that $x - 3$ is negative and so its absolute value is $-(x - 3)$.)

c) $\lim_{x \to 0} \frac{\sin(5x^2)}{3x} = \lim_{x \to 0} \frac{\sin(5x^2)}{5x^2} \frac{5x^2}{3x} = \lim_{x \to 0} \frac{\sin(5x^2)}{5x^2} \frac{5x}{3} = 1(0) = 0$.

d) $\lim_{x \to 1} \frac{1 - \sqrt{x}}{x^2 - 6x + 5} = \lim_{x \to 1} \frac{1 - x}{(1 + \sqrt{x})(x - 1)(x - 5)} = \lim_{x \to 1} \frac{-1}{(1 + \sqrt{x})(x - 5)} = \frac{1}{8}$. (There was a sign error in the original version of this file.)

2 (25 points) Let $f(x)$ be a function.

a) *Express the definition of the derivative $f'(x)$ in the form* $\lim_{h \to 0} \dots$.

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

b) *Express the definition of the devivative $f'(x)$ in the form* $\lim_{t \to x} \dots$.

$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$.

c) *Use* **one of the definitions above** *to compute $f'(x)$ if* $f(x) = \frac{1}{4x}$.

$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{\frac{1}{4t} - \frac{1}{4x}}{t - x}$
$= \lim_{t \to x} \frac{4x - 4t}{(4x)(4t)(t - x)} = \lim_{t \to x} \frac{-4}{(4x)(4t)} = \frac{-1}{4x^2}$.

d) Find an equation of the line tangent to the curve of Part c) at $x = 2$.

The slope of the tangent line is $f'(2) = \frac{-1}{16}$ and it passes through the point $(2, f(2)) = (2, \frac{1}{8})$. Thus it has equation $y = \frac{1}{8} - \frac{1}{16}(x - 2)$.

3 (20 points) The height y (in feet) of a ball t seconds after it is thrown up in the air is given by the formula $y = f(t) = 80t - 16t^2$.

a) *What is the ball's average velocity during its first two seconds of flight ?*

This does not involve a limit; it is the difference quotient $\frac{f(2)-f(0)}{2-0} = \frac{160-64}{2} = 48$ ft/sec.

b) *What is the instantaneous velocity of the ball at $t = 2$?*

$f'(t) = 80 - 32t$ so $f'(2) = 16$ ft/sec.

c) *When is the ball 24 feet high ?*

This is when $f(t) = 24$, i.e., $-16t^2 + 80t = 24$ which has two solutions: $t = 5 \pm \sqrt{19}$ seconds –once on the way up and once on the way down. (Sorry for the numbers.)

d) *What is the maximum height achieved by the ball ?*

This happens when $f'(t) = 0$, i.e., when $t = 2.5$ sec. Thus the maximum height is $f(2.5) = 100$ ft.

4 (25 points) *Give an explicit function [like $f(x) = 3x$] that exemplifies each phenomenon. No explanation is required.*

a) *a function that is* **not a polynomial**, *but is continuous on its domain*
b) *a function having a removable discontinuity*
c) *a function whose graph has the line $y = \frac{5}{2}$ as a horizontal asymptote*
d) *a function whose graph has two vertical asymptotes*
e) *a fucntion whose graph has a slant asymptote.*

a) $f(x) = \frac{1}{1+x^2}$
b) $f(x) = \frac{x}{x}$
c) $f(x) = \frac{5}{2} + \frac{1}{x}$
d) $f(x) = \frac{1}{x^2-4}$
e) $f(x) = 3x + \frac{1}{x}$.

Many other answers are possible.

5 (10 points) *State the Intermediate Value Theorem and use it to show that the equation $2^x = 3x^2$ has a solution in the interval $[-1, 0]$. You must include appropriate prose for full credit.*

a) Suppose $f(x)$ is a continuous function defined on a closed interval $[a, b]$. If $f(a)$ and $f(b)$ have opposite signs, then there is some number c between a and b satisfying $f(c) = 0$.

b) Set $f(x) := 2^x - 3x^2$. Then $f(x)$ is continuous everywhere, and $f(-1) = \frac{1}{2} - 3$ is negative, while $f(0) = 1$ is positive. By Part a), there is a number c in $[-1, 0]$ satisfying $f(c) = 0$, i.e., $2^c = 3c^2$.