JOINTLY QUASINORMAL FAMILIES ARE REFLEXIVE

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Abstract. We show that each jointly quasinormal family of operators acting on a separable Hilbert space generates a reflexive algebra with property $\mathbb{A}_1(1)$.

In [9], D. Sarason proved that each commuting family of normal operators is reflexive. Doubly commuting pairs of isometries were shown to be reflexive in [8]. Recently, H. Bercovici [2] and W.S. Li–J. McCarthy [6], applying results of K. Horák–V. Müller [5], have extended the latter result to arbitrary commuting families of isometries. In this note, we observe that direct integral theory leads to a common generalization of the normal and isometric cases.

All operators discussed in this note are bounded and act on separable Hilbert spaces. Let $\mathcal{S}$ be a family of operators acting on a common Hilbert space. Then we write $\mathcal{W}(\mathcal{S})$ for the weakly closed algebra generated by $\mathcal{S}$ and the identity operator; $\mathcal{S}$ is *reflexive* if each operator leaving invariant all subspaces, which are invariant under all members of $\mathcal{S}$, must belong to $\mathcal{W}(\mathcal{S})$. An algebra of operators $\mathcal{A}$ has property $\mathbb{A}_1(1)$ if given a weak* continuous linear functional ϕ on $L(H)$ and $\varepsilon > 0$, there are $a, b \in H$ such that $\|a\| \cdot \|b\| \leq (1+\varepsilon)\|\phi\|$ and $\phi(A) = (A\,a, b)$ for all $A \in \mathcal{A}$.

An operator T is called *quasinormal* if T commutes with T^*T. W. Wogen [11] proved that individual quasinormal operators are reflexive. Following A. Lubin [7], we call a commutative family $\mathcal{S}$ of operators *jointly quasinormal* if S and T^*T commute for any $S, T \in \mathcal{S}$. Commutative families of normal operators or isometries are automatically jointly quasinormal, as are doubly commuting families of quasinormal operators. Example 2 below exhibits a commuting pair of quasinormal operators which is not jointly quasinormal.

Theorem 1. *Every jointly quasinormal family $\mathcal{S}$ of operators is reflexive and has property $\mathbb{A}_1(1)$.*

Proof. Since the underlying Hilbert space is separable, we may as well assume the family $\mathcal{S}$ to be countable. Write $\mathcal{Z}$ for the commutative von Neummann algebra generated by $\{T^*T : T \in \mathcal{S}\}$. By direct integral theory (see [10]) $\mathcal{Z}$ is the diagonal algebra corresponding to a direct integral decomposition of the underlying Hilbert space $\mathcal{H} = \int_\Lambda^\oplus H(\lambda)\,d\mu(\lambda)$. Here μ is a finite regular Borel measure on Λ and we have $\mathcal{Z} = \int_\Lambda^\oplus \mathcal{Z}(\lambda)\,d\mu(\lambda)$, where each $\mathcal{Z}(\lambda)$ consists of scalar multiples of $I_{H(\lambda)}$. For simplicity of notation, we assume that $H(\lambda) \equiv H$ and the corresponding field of measurable vectors is the constant field. Each

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$T \in \mathcal{S}$ is decomposable, and by our choice of $\mathcal{Z}$, we know that $T^*(\lambda)T(\lambda)$ is a scalar multiple of the identity for almost all λ. Discarding a set of measure zero if necessary, we can assume $T(\lambda)$ to be a scalar multiple of an isometry for each $T \in \mathcal{S}$ and $\lambda \in \Lambda$.

For each fixed λ, the algebra $\mathcal{W}(T(\lambda) : T \in \mathcal{S})$ is generated by a family of commuting isometries . By [2, Theorems 2.3–2.4], it is reflexive and has property $\mathbb{A}_1(1)$. Hence, by [1, Proposition 5.6], $\mathcal{W}(\mathcal{Z} \cup \mathcal{S})$ is reflexive, and by [4, Theorem 3.6], has property $\mathbb{A}_1(1)$. Thus its subalgebra $\mathcal{W}(\mathcal{S})$ is reflexive and has property $\mathbb{A}_1(1)$ by [4, Proposition 2.5].

Example 2. Write U for the forward bilateral shift relative to an orthonormal basis $\{e_n\}_{n=-\infty}^{\infty}$. Denote by P and Q the projections onto the spaces spanned by $\{e_n : n \geq 0\}$ and $\{e_n : n \leq 0\}$ respectively. Set $S = UP$ and $T = U^*Q$. The quasinormality of S and T follows from the fact that P and Q are invariant under U and U^* respectively. It is easy to check that $ST = 0 = TS$. On the other hand, while $T^*TS = QUP = 0$, we have $ST^*T = UPQ \neq 0$. Thus $\{S, T\}$ is a commuting pair of quasinormal operators which is not jointly quasinormal.

Remark. Various notions of "joint near normality" have been compared in the literature. See for example R. Curto [3] and the references cited there. Example 2 fits into this setting; we mention some of its relevant features:

(1) the commuting generators S and T are quasinormal,
(2) every operator in $\mathcal{W}(S, T)$ is hyponormal,
(3) the pair S, T is not jointly subnormal or even jointly weakly hyponormal.

Remark. It is conceivable that Theorem 1 remains true for arbitrary commutative families of quasinormal operators. In particular, one can see that the operators in Example 2 generate a reflexive algebra with Property $\mathbb{A}_1(1)$.

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