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COMPACT OPERATORS IN REDUCTIVE ALGEBRAS

EDWARD A. AZOFF

Let $\mathscr{H}$ be a Hilbert space and denote the collection of (bounded, linear) operators on $\mathscr{H}$ by $\mathscr{L}(\mathscr{H})$. Throughout this paper, the term 'algebra' will refer to a subalgebra of $\mathscr{L}(\mathscr{H})$; unless otherwise stated, it will not be assumed to contain I or to be closed in any topology.

An algebra is said to be *transitive* if it has no non-trivial invariant subspaces. The following lemma has revolutionized the study of transitive algebras. For a proof and a general discussion of its implications, the reader is referred to [5].

LEMMA 1 (Lomonosov). *Suppose $\mathfrak{A}$ is a transitive algebra and K is a non-zero compact operator. Then there exists an $A \in \mathfrak{A}$ such that the operator AK has* 1 *as an eigenvalue.*

COROLLARY 2 [5]. *Let $\mathfrak{A}$ be a transitive algebra containing a non-zero compact operator. Suppose moreover that $\mathfrak{A}$ is weakly closed and contains I. Then $\mathfrak{A} = \mathscr{L}(\mathscr{H})$.*

The purpose of this paper is to prove a generalization of this corollary. To describe it, we first need two definitions.

Definition [6]. *An algebra $\mathfrak{A}$ is called reductive if it is weakly closed and every invariant subspace for $\mathfrak{A}$ reduces $\mathfrak{A}$.*

Definition. Let $\mathfrak{A}$ be a reductive algebra and denote by $\mathscr{B}$, the von Nuemann algebra generated by $\mathfrak{A}$. Then for $A \in \mathfrak{A}$, we define the *central support* of A to be the smallest (self-adjoint) projection P in the center of $\mathscr{B}$ such that $AP = A$.

THEOREM 3. *Let $\mathfrak{A}$ be a reductive algebra containing a compact operator K. Then the central support P of K belongs to $\mathfrak{A}$ and $P\mathfrak{A}P$ is self-adjoint.*

Before embarking on the proof of the Theorem, it seems appropriate to make several observations. First, note that Theorem 3 contains Corollary 2 as a special case. Indeed, in a transitive algebra, every non-zero operator has central support I, and the von Neumann double-commutant theorem assures us that $\mathscr{L}(\mathscr{H})$ is the only transitive von Neumann algebra.

In fact, Theorem 3 is, in a sense, the best result one could hope for. This is because $(I - P)\mathfrak{A}(I - P)$ is a reductive algebra about which we know nothing (i.e., it could be any reductive algebra).

Finally, we single out two corollaries of Theorem 3. Corollary 5 was pointed out to the author by Frank Gilfeather.

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COROLLARY 4 [7]. *Suppose $\mathfrak{A}$ is a reductive algebra containing an injective compact operator K. Then $\mathfrak{A}$ is self-adjoint.*

Proof. The central support of an injective operator is I.

COROLLARY 5. *Let $\mathfrak{A}$ be a reductive algebra and suppose the supremum of the central supports of the compact operators in $\mathfrak{A}$ is I. Then $\mathfrak{A}$ is self-adjoint.*

Proof. Let $\mathscr{B}$ be the von Neumann algebra generated by $\mathfrak{A}$. Applying Zorn's lemma, we find a maximal orthogonal family $\mathscr{P}$ of central projections in $\mathscr{B}$ such that $\mathscr{B}P \subseteq \mathfrak{A}$ for each $P \in \mathscr{P}$. Let $P_0 = \sum \mathscr{P}$. Then the weak closure of $\mathfrak{A}$ shows that $\mathscr{B}P_0 \subseteq \mathfrak{A}$. Note that if $I \neq P_0$, then $\mathfrak{A}(I - P_0)$ would contain a non-zero compact operator. In view of the theorem, this contradicts the maximality of $\mathscr{P}$. Thus $I = P_0$ and the proof is complete.

In proving Theorem 3, it will be convenient to isolate two lemmas. Lemma 6 is a slight variation of Corollary 2 and its proof uses several arguments found in [5].

LEMMA 6. *Let $\mathfrak{A}$ be a transitive algebra and $\mathscr{J}$ a norm closed, two-sided ideal in $\mathfrak{A}$. Suppose $\mathscr{J}$ contains a non-zero compact operator. Then $\mathscr{J}$ contains all compact operators.*

Proof. Let K be a non-zero compact operator in $\mathscr{J}$. By Lomonosov's lemma, there exists an $A \in \mathfrak{A}$ such that AK belongs to $\mathscr{J}$ and has a fixed point. Note that the span of $\mathscr{J}$ and I is a Banach algebra. Thus by applying an appropriate analytic function to AK, we find a non-zero finite rank idempotent J in the span of $\mathscr{J}$ and I. In fact, there is a sequence of polynomials $\{p_n\}$ for which $p_n(AK) \to J$. Since the distance from I to the compacts is 1, we conclude that $p_n(0) \to 0$ and hence that J actually belongs to $\mathscr{J}$.

Note that $J\mathscr{J}J|_{\mathrm{Ran}\ J} = J\mathfrak{A}J|_{\mathrm{Ran}\ J}$ is a subalgebra of $\mathscr{L}(\mathrm{Ran}\ J)$. Since $J\mathfrak{A}J$ acts transitively on $\mathrm{Ran}\ J$ we conclude that $J\mathscr{J}J|_{\mathrm{Ran}\ J} = \mathscr{L}(\mathrm{Ran}\ J)$ (Burnside's theorem). In particular $J\mathscr{J}J$ (and hence $\mathscr{J}$) contains a rank one operator. The lemma now follows by the transitivity of $\mathfrak{A}$.

LEMMA 7. *Let $\mathfrak{A}$ be a reductive algebra and suppose J is a finite rank idempotent in $\mathfrak{A}$. Then $J\mathfrak{A}J|_{\mathrm{Ran}\ J}$ is self-adjoint.*

Proof. Suppose M is invariant under $J\mathfrak{A}J|_{\mathrm{Ran}\ J}$. Then $(\mathfrak{A}M)^-$ is invariant under $\mathfrak{A}$ and $J(\mathfrak{A}M)^- = M$. The algebra $\mathfrak{A}$ being reductive, we see that $(\mathfrak{A}M)^\perp$ is also invariant under $\mathfrak{A}$. Since $J \in \mathfrak{A}$, it follows that $J(\mathfrak{A}M)^\perp$ is contained in $(\mathfrak{A}M)^\perp$ and hence $J(\mathfrak{A}M)^\perp \subseteq M^\perp$. Thus $\mathrm{Ran}\ J$ is the orthogonal direct sum of M and $J(\mathfrak{A}M)^\perp$. Since $J(\mathfrak{A}M)^\perp$ is invariant under $J\mathfrak{A}J|_{\mathrm{Ran}\ J}$, we see that $J\mathfrak{A}J|_{\mathrm{Ran}\ J}$ is reductive.

This completes the proof since it is known ([1, p. 127, Theorem 4] or [6, Theorem 2]) that a reductive algebra acting on a finite-dimensional space must be self-adjoint.

Proof of Theorem 3. Let $\mathscr{V}$ be the von Neumann algebra generated by $\mathfrak{A}$. We are going to show there is a non-zero self-adjoint projection $Q \leqq P$ in the center of $\mathscr{V}$ such that $\mathscr{V}Q \subseteq \mathfrak{A}$. This will complete the proof since a standard maximality argument then gives $\mathscr{V}P \subset \mathfrak{A}$, i.e., $P \in \mathfrak{A}$ and $P\mathfrak{A}P = \mathscr{V}P$.

Consider the von Neumann algebra $\mathscr{B} = \mathscr{V}|_{\text{Ran } P}$. Note that the central support of $K|_{\text{Ran } P}$ in $\mathscr{B}$ is I. Applying [4, Proposition 1], we conclude that the center of $\mathscr{B}$ is atomic. Let Q be a minimal central projection in $\mathscr{B}$. Since $K|_{\text{Ran } Q}$ is non-zero, it follows that $\mathscr{B}|_{\text{Ran } Q}$ is a type 1 factor.

We now apply [2, Corollary 3, p. 124] to $\mathscr{B}|_{\text{Ran } Q}$. Thus we find Hilbert spaces $\mathscr{M}$ and $\mathscr{N}$ such that $\mathscr{B}|_{\text{Ran } Q}$ is unitarily equivalent to $\mathscr{L}(\mathscr{M}) \otimes C_{\mathscr{N}}$. The compactness of $K|_{\text{Ran } Q}$ shows that $\mathscr{N}$ must be finite-dimensional. In the sequel, we will identify $\mathscr{M}$ and $\mathscr{N}$ with subspaces of Ran Q.

Note that $\mathfrak{A}$ and $\mathscr{V}$ have the same invariant subspaces. Since $\mathscr{M}$ reduces $\mathscr{V}$, the same is true of $\mathfrak{A}|_{\mathscr{M}}$ and $\mathscr{V}|_{\mathscr{M}}$. Thus $\mathfrak{A}|_{\mathscr{M}}$ is a transitive subalgebra of $\mathscr{L}(\mathscr{M})$ containing the non-zero compact operator $K|_{\mathscr{M}}$. By Lomonosov, there exists an $A \in \mathfrak{A}$ such that $A|_{\mathscr{M}}K|_{\mathscr{M}}$ has a fixed point. Taking an appropriate analytic function of AK, we find a finite rank idempotent J in $\mathfrak{A}$ for which $J|_{\mathscr{M}} \neq 0$.

The restricted algebras $J\mathfrak{A}J|_{\text{Ran } J}$ and $J\mathscr{V}J|_{\text{Ran } J}$ have the same invariant subspaces. Moreover, by Lemma 7, they are both self-adjoint. Thus by the double-commutant theorem, they coincide. In particular, QJ is a non-zero, finite-rank operator in $J\mathfrak{A}J$ (and hence in $\mathfrak{A}$) supported on Ran Q.

Denote by $\mathscr{C}$ the collection $\{A \in \mathfrak{A} | A \text{ is supported on Ran } Q\}$ and set $\mathscr{I} = \mathscr{C}|_{\mathscr{M}}$. Then $\mathscr{I}$, considered as a two-sided ideal over $\mathfrak{A}|_{\mathscr{M}}$ satisfies the hypothesis of Lemma 6. Thus $\mathscr{I} = \mathscr{L}(\mathscr{M})$ ($\mathscr{I}$ is weakly closed) and hence $\mathscr{C} = \mathscr{V}Q$. This shows $\mathscr{V}Q \subseteq \mathfrak{A}$ and completes the proof of the Theorem.

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University of Georgia,
Athens, Georgia