K-REFLEXIVITY IN FINITE DIMENSIONAL SPACES

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Let $\mathcal{H}$ be an n-dimensional Hilbert space, $n \geq 4$. A subalgebra $\mathfrak{A}$ of $\mathcal{L}(\mathcal{H})$ is defined to be *k-reflexive* if $\mathfrak{A}^{(k)}$, a k-fold copy of $\mathfrak{A}$, is reflexive. The main theorems of the paper state that (1) every subalgebra of $\mathcal{L}(\mathcal{H})$ is $(n - 1)$-reflexive and (2) every commutative subalgebra of $\mathcal{L}(\mathcal{H})$ is $(n/2)$-reflexive. Examples are given to show these results are "best-possible".

Several alternate characterizations of k-reflexivity provide the main technique of the paper. As a by-product, we are able to exhibit a commutative algebra $\mathfrak{A}$ such that $\mathfrak{A} \neq \mathfrak{A}'' \cap \text{Alg Lat } \mathfrak{A}$; this answers a question of P. Rosenthal.

1. Introduction. All Hilbert spaces discussed in this paper will be complex and finite dimensional. We denote by $\mathcal{L}(\mathcal{H})$ the collection of all (linear) operators on the Hilbert space $\mathcal{H}$. Let $\mathfrak{A}$ be an identity-containing subalgebra of $\mathcal{L}(\mathcal{H})$. Then *Lat* $\mathfrak{A}$ denotes the collection of subspaces of $\mathcal{H}$ invariant under each operator in $\mathfrak{A}$ and *Alg Lat* $\mathfrak{A}$ denotes the collection of operators in $\mathcal{L}(\mathcal{H})$ leaving each subspace in Lat $\mathfrak{A}$ invariant. Finally, for $A \in \mathcal{L}(\mathcal{H})$ we write $A^{(k)}$ for the direct sum of k copies of A and $\mathfrak{A}^{(k)}$ for $\{A^{(k)} \mid A \in \mathfrak{A}\}$.

A subalgebra $\mathfrak{A}$ of $\mathcal{L}(\mathcal{H})$ is said to be *reflexive* if it is determined by its invariant subspaces, i.e., $\mathfrak{A} = \text{Alg Lat } \mathfrak{A}$. In [3] Deddens and Fillmore gave necessary and sufficient conditions for $\mathfrak{A}_A$, the algebra generated by A (and the identity), to be reflexive. The more general problem of determining all reflexive subalgebras of $\mathcal{L}(\mathcal{H})$ has so far defied solution.

The purpose of this paper is to measure how nonreflexive an algebra can be. More precisely, we call an algebra $\mathfrak{A}$ *k-reflexive* if $\mathfrak{A}^{(k)}$ is reflexive. Let $\mathcal{H}$ be n-dimensional, $n \geq 3$. Our main results follow.

(1) There exists a subalgebra of $\mathcal{L}(\mathcal{H})$ which is not $(n - 2)$-reflexive.

(2) Every subalgebra of $\mathcal{L}(\mathcal{H})$ is $(n - 1)$-reflexive.

These results are obtained in Section 3; for the proofs, we rely on several alternate characterizations of k-reflexivity obtained in Section 2.

The final three sections of the paper are devoted to extensions of the above results. In Section 4 we present analogues of (1) and (2) for commutative subalgebras of $\mathcal{L}(\mathcal{H})$; as a corollary, we see that not every commutative algebra $\mathfrak{A}$ satisfies $\mathfrak{A} = \mathfrak{A}'' \cap \text{Alg Lat } \mathfrak{A}$, thus answering a question of [6]. A generalization of (2) is proved in Section 5, and the paper closes with a discussion of vector spaces over fields other than **C**.

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2. Fundamental equivalences. We begin with several remarks concerning reflexivity.

Example 2.1. Write M_n for $\mathcal{L}(\mathbf{C}_n)$. We agree to identify an operator $A \in M_n$ with its matrix relative to the standard basis $e_1, \cdots, e_n$. Set $\mathfrak{A} = \left\{\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \in M_2\right\}$. Then $\mathfrak{A}$ is a (commutative) nonreflexive algebra.

Proposition 2.2. *For each integer k, $\mathcal{L}(\mathcal{H})$ is k-reflexive.*

Proof. Suppose $\mathrm{Lat}\, \mathcal{L}(\mathcal{H})^{(k)} \subseteq \mathrm{Lat}\, B$. Then the subspaces $\mathcal{H} \oplus 0 \cdots \oplus 0, \cdots, 0 \oplus \cdots \oplus 0 \oplus \mathcal{H}$ are invariant under B and we have $B = B_1 \oplus \cdots \oplus B_k$ for some $B_1 \cdots B_k \in \mathcal{L}(\mathcal{H})$. Moreover, since the subspace $S = \{x \oplus \cdots \oplus x \mid x \in \mathcal{H}\}$ is invariant under $\mathcal{L}(\mathcal{H})^{(k)}$, we have $B_1 x = \cdots = B_k x$ for each $x \in \mathcal{H}$. Thus $B = B_1^{(k)} \in \mathcal{L}(\mathcal{H})^{(k)}$ and the proof is complete.

Corollary 2.3. *The following are equivalent.*

(1) *$\mathfrak{A}$ is k-reflexive.*

(2) *Any T (in $\mathcal{L}(\mathcal{H})$) satisfying* $\mathrm{Lat}\, \mathfrak{A}^{(k)} \subseteq \mathrm{Lat}\, T^{(k)}$ *must belong to $\mathfrak{A}$.*

Proof. (1) ⇒ (2). By hypothesis $\mathfrak{A}^{(k)}$ is reflexive. Hence $\mathrm{Lat}\, \mathfrak{A}^{(k)} \subseteq \mathrm{Lat}\, T^{(k)}$ implies $T^{(k)} \in \mathfrak{A}^{(k)}$, i.e., $T \in \mathfrak{A}$.

(2) ⇒ (1). Suppose $\mathrm{Lat}\, \mathfrak{A}^{(k)} \subseteq \mathrm{Lat}\, S$. Then $\mathrm{Lat}\, \mathcal{L}(\mathcal{H})^{(k)} \subseteq \mathrm{Lat}\, S$. But $\mathcal{L}(\mathcal{H})^{(k)}$ is reflexive so $S = T^{(k)}$ for some $T \in \mathcal{L}(\mathcal{H})$. Applying (2), we see $T \in \mathfrak{A}$, i.e., $S \in \mathfrak{A}^{(k)}$.

The next proposition provides several conditions equivalent to (2) of Corollary 2.3. It provides the basic tool for this paper.

Proposition 2.4. *Let $T \in \mathcal{L}(\mathcal{H})$. Then the following are equivalent.*

(1) $\mathrm{Lat}\, \mathfrak{A}^{(k)} \subseteq \mathrm{Lat}\, T^{(k)}$.

(2) *For each $x \in \mathcal{H}^{(k)}$, $T^{(k)}x \in \mathfrak{A}^{(k)}x$.*

(3) *For each $x_1, \cdots, x_k, y_1, \cdots, y_k \in \mathcal{H}$, if $\sum_{i=1}^{k} (Ax_i, y_i) = 0$ for all $A \in \mathfrak{A}$, then $\sum_{i=1}^{k} (Tx_i, y_i) = 0$.*

Proof. (1) ⇒ (2). The subspace $\mathfrak{A}^{(k)}x$ is invariant under $\mathfrak{A}^{(k)}$ and thus under $T^{(k)}$. But $x \in \mathfrak{A}^{(k)}x$ (since $I \in \mathfrak{A}$) and thus $T^{(k)}x \in \mathfrak{A}^{(k)}x$.

(2) ⇒ (1). Let M be invariant under $\mathfrak{A}^{(k)}$ and suppose $x \in M$. Then $T^{(k)}x \in \mathfrak{A}^{(k)}x \subseteq M$.

(2) ⇔ (3). $T^{(k)}x \in \mathfrak{A}^{(k)}x$ if and only if any vector (in $\mathcal{H}^{(k)}$) orthogonal to $\mathfrak{A}^{(k)}x$ is also orthogonal to $T^{(k)}x$.

Corollary 2.5. *Suppose $\mathfrak{A}$ is k-reflexive. Then $\mathfrak{A}$ is l-reflexive for every $l \geq k$.*

Proof. This follows immediately from the equivalence (1) ⇔ (2) of the preceding proposition.

Corollary 2.6. *Every subalgebra of M_n is n-reflexive.*

Proof. Suppose Lat $\mathfrak{A}^{(n)} \subseteq$ Lat $T^{(n)}$. Applying (2) of Proposition 2.4 we find an $A \in \mathfrak{A}$ such that $Ae_1 = Te_1 \cdots Ae_n = Te_n$. But then $T = A$.

3. The main result.

The main object of this section is to prove the following improvement of Corollary 2.6.

THEOREM 3.1 *Suppose $n \geq 3$. Then every subalgebra of M_n is $(n-1)$-reflexive.*

Example 2.1 shows the hypothesis $n \geq 3$ is essential. The next example shows Theorem 3.1 is the "best-possible" result.

Example 3.2. There exists a subalgebra of M_n which is not $(n-2)$-reflexive.

Proof. Take $\mathfrak{A} = \{A = [a_{ij}] \in M_n \mid A$ is upper triangular, A has a single eigenvalue, and the supertrace of A is $0\}$. It is easily checked that $\mathfrak{A}$ is an algebra. Let $T \in M_n$ be given by $Tx = (x, e_2)e_1$. (The matrix of T has a single nonzero entry—a 1 in the first row, second column.) We are going to show $\mathfrak{A}$ is not $(n-2)$-reflexive by showing that Lat $\mathfrak{A}^{(n-2)} \subseteq$ Lat $T^{(n-2)}$. Indeed let $x_1, \cdots, x_{n-2}$ be any vectors in $\mathbf{C}_n$. Find a nonzero vector $y \in \mathbf{C}_n$ satisfying

(a) $y \perp x_1, \cdots, y \perp x_{n-2}$,
(b) $y \perp e_1$, and
(c) the first nonzero coordinate of y (call it the k-th) is 1.

We construct a matrix C by setting the $(k-1)$-st row of C equal to $\bar{y}$ (the vector whose coordinates are the conjugates of the coordinates of y) and all other rows of C equal to 0. Then $(T - C) \in \mathfrak{A}$ and $(T - C)x_i = Tx_i$, $i = 1, \cdots, n-2$. By Proposition 2.4 (2), Lat $\mathfrak{A}^{(n-2)} \subseteq$ Lat $T^{(n-2)}$.

We now begin working towards a proof of Theorem 3.1. Our main tool will be Proposition 2.4 (3); in order to use it effectively, we introduce the following terminology. Fix $\mathfrak{K}$. By a *relation* R, we mean an expression of the form $\sum_{i=1}^{k} ((\cdot)x_i, y_i) = 0$ where $x_1, \cdots, x_k, y_1, \cdots, y_k \in \mathfrak{K}$; k is called the *order* of the relation. An operator $A \in \mathfrak{L}(\mathfrak{K})$ is said to *satisfy* R if $\sum_{i=1}^{k} (Ax_i, y_i) = 0$; a subalgebra $\mathfrak{A} \subseteq \mathfrak{L}(\mathfrak{K})$ is said to *satisfy* R if each $A \in \mathfrak{A}$ satisfies R. Finally, by a *characterization* for $\mathfrak{A}$, we mean a set $\mathfrak{C}$ of relations which are simultaneously satisfied by an operator if and only if it belongs to $\mathfrak{A}$; the maximal order of the relations in $\mathfrak{C}$ (if it exists) is called the *order* of the characterization.

Let $A = [a_{ij}] \in M_n$. Suppose for example that $5a_{11} + 3a_{12} = 0$. This is the same as saying

(1) $5(Ae_1, e_1) + 3(Ae_2, e_1) = 0$ or
(2) $(Ae_1, 5e_1) + (Ae_2, 3e_1) = 0$ or
(3) $(A(5e_1 + 3e_2), e_1) = 0$.

The latter two statements are relations of orders 2 and 1 respectively. General remarks of this type can be made for any linear constraint on the entries of A. Since every subalgebra of M_n is a submanifold of $\mathbf{C}_{n^2}$, it follows that every algebra $\mathfrak{A} \subseteq M_n$ has at least one characterization (in fact there are infinitely many).

LEMMA 3.3. *Let $\mathfrak{A}$ be a subalgebra of $\mathfrak{L}(\mathfrak{K})$. Then $\mathfrak{A}$ is k-reflexive if and only if $\mathfrak{A}$ has a characterization of order k.*

Proof. ($\Rightarrow$) Let $\mathcal{C}$ be the collection of all relations of order less than or equal to k satisfied by $\mathfrak{A}$. Then by Proposition 2.4 (3) $\mathcal{C}$ is a characterization of $\mathfrak{A}$.

($\Leftarrow$). Let $\mathcal{C}$ be a characterization of order k for $\mathfrak{A}$ and suppose Lat $\mathfrak{A}^{(k)} \subseteq$ Lat $T^{(k)}$. By Proposition 2.4 (3), every relation in $\mathcal{C}$ is satisfied by T. Hence $T \varepsilon \mathfrak{A}$.

We say two relations are *equivalent* if any operator satisfying one satisfies the other.

LEMMA 3.4. *Suppose rank $\langle x_1, \cdots, x_n \rangle \leq k$, $k \neq 0$. Then $\sum_{i=1}^n ((\cdot)x_i, y_i) = 0$ is equivalent to a relation of order k.*

Proof. If $k \geq n$, there is nothing to show. Otherwise, one of the x_i (say it's x_n) is a linear combination of the others, $x_n = \sum_{i=1}^{n-1} \lambda_i x_i$. Substitution for x_n and a little elementary algebra yields $\sum_{i=1}^n (Ax_i, y_i) = \sum_{i=1}^{n-1} (Ax_i, y_i + \bar{\lambda}_i y_n)$. Thus the given relation may be replaced by a relation of order $(n - 1)$. An easy induction argument now completes the proof.

Remark 3.5. A similar result holds if rank $\langle y_1, \cdots, y_n \rangle \leq k$.

THEOREM 3.6. *Let $\mathfrak{A} \subseteq M_n$ and suppose $\mathfrak{A}$ contains an operator, none of whose eigenvalues has (algebraic) multiplicity strictly greater than k. Then $\mathfrak{A}$ is k-reflexive.*

Proof. Choose $A_0 \varepsilon \mathfrak{A}$, none of whose eigenvalues has multiplicity strictly greater than k. Note that k-reflexivity is preserved under similarity so there is no loss of generality in assuming A_0 is in Jordan canonical form. Now by taking appropriate polynomials in A_0, we see that the idempotents corresponding to the various eigenvalue blocks of A_0 also belong to $\mathfrak{A}$. Let these idempotents be $P_1, \cdots, P_r$. Then $P_1 + \cdots + P_r = I$ and rank $P_i \leq k$ for each i.

Let $\mathcal{C}$ be a characterization of $\mathfrak{A}$. Suppose $R : \sum_{i=1}^m ((\cdot)x_i, y_i) = 0$ is a relation in $\mathcal{C}$. Then the relations $R_j : \sum_{i=1}^m ((\cdot)P_j x_i, y_i) = 0$, $j = 1, \cdots, r$, are also satisfied by $\mathfrak{A}$ since $P_j \varepsilon \mathfrak{A}$. Moreover any operator satisfying $R_1, \cdots, R_r$ must also satisfy R. In other words, we may replace R by $R_1, \cdots, R_r$ in $\mathcal{C}$. But rank $\langle P_j x_1, \cdots, P_j x_m \rangle \leq$ rank $P_j \leq k$. Hence by Lemma 3.4, each R_j may be recast as a relation of order k. This shows $\mathfrak{A}$ has a characterization of order less than or equal to k and thus (Lemma 3.3) $\mathfrak{A}$ is k-reflexive.

Proof of Theorem 3.1. Let $\mathfrak{A}$ be a subalgebra of M_n. If some operator in $\mathfrak{A}$ has more than one eigenvalue, we're done by the previous theorem. Thus we may assume every operator in $\mathfrak{A}$ is a scalar plus a nilpotent. It follows from Engel's theorem [5; p. 17] that $\mathfrak{A}$ is similar to an algebra of upper-triangular matrices. Without loss of generality we assume $\mathfrak{A}$ is of this form. Suppose each $A \varepsilon \mathfrak{A}$ satisfies $R_0 : \sum_{i,j=1}^{n} \lambda_{ij}(Ae_j , e_i) = 0$. Then A also satisfies $R_1 : (Ae_j , e_i) = 0, i > j$ (since A is in upper-triangular form), $R_2 : (Ae_i , e_i) - (Ae_j , e_j) = 0$ (since A has a single eigenvalue), and $R_3 : \sum_{i \leq j} \lambda_{ij}(Ae_j , e_i) = 0$. Conversely, an A satisfying R_1 , R_2 , R_3 also satisfies R_0 . Moreover, the term $\lambda_{nn}(Ae_n , e_n)$ in R_3 can be replaced by $\lambda_{nn}(Ae_1 , e_1)$ and thus R_3 can be recast in the form $\sum_{i=1}^{n-1} (Ay_i , e_i) = 0$, a relation of order $n - 1$. Thus $\mathfrak{A}$ has a characterization of order $n - 1$ and we're done by Lemma 3.3.

Remark 3.7. As Example 2.1 shows, the hypothesis $n \geq 3$ in the previous theorem is essential. Indeed R_2 (above) is a relation of order 2.

4. Commutative algebras. In this section, we examine how Theorem 3.1 can be improved if one restricts attention to commutative algebras. The situation is described by the following theorem.

THEOREM 4.1. *Suppose $n \geq 4$. Then every commutative subalgebra of M_n is $(n/2)$-reflexive.*

Here, and throughout this section, $n/2$ is to be interpreted as the greatest integer in $n/2$. Example 2.1 shows the restriction $n \geq 4$ is necessary. The next example shows Theorem 4.1 is "best-possible".

Example 4.2. There exists a commutative subalgebra of M_n which is not $(n/2 - 1)$-reflexive.

Proof. For purposes of simplicity, we assume $n = 2m$ is even. Set

$$\mathfrak{A} = \left\{ \begin{bmatrix} \lambda I & D \\ 0 & \lambda I \end{bmatrix} \,\middle|\, D \varepsilon M_m \text{ has trace zero} \right\}.$$

Let $B = [b_{ij}]$ in M_n have $b_{1m+1} = 1$ and all other entries zero. Using a construction similar to that of Example 3.2, one sees that for any vectors $x_1 , \cdots , x_{m-1}$ in $\mathbf{C}_n$ there is an $A \varepsilon \mathfrak{A}$ satisfying $Ax_i = Bx_i$, $i = 1, \cdots , m - 1$. Hence Lat $\mathfrak{A}^{(m-1)} \subseteq$ Lat $B^{(m-1)}$ and $\mathfrak{A}$ is not $(m - 1)$-reflexive.

If $\mathfrak{A}$ is a subalgebra of $\mathcal{L}(\mathcal{H})$, we write $\mathfrak{A}'$ for the commutant of $\mathfrak{A}$, i.e., $\mathfrak{A}' = \{B \varepsilon \mathcal{L}(\mathcal{H}) \mid AB = BA \text{ for all } A \varepsilon \mathfrak{A}\}$. The next lemma will be used in the proof of Theorem 4.1.

LEMMA 4.3. *Suppose* Lat $\mathfrak{A}^{(2)} \subseteq$ Lat $T^{(2)}$. *Then $T \varepsilon \mathfrak{A}''$. If $\mathfrak{A}$ consists of operators with a single eigenvalue, then T also has a single eigenvalue.*

Proof. Let $x \varepsilon \mathcal{H}$ and $B \varepsilon \mathfrak{A}'$. By Proposition 2.4(2), there exists an $A \varepsilon \mathfrak{A}$ such that $Ax = Tx$ and $A(Bx) = T(Bx)$. But then $BTx = BAx = ABx = TBx$ so B commutes with T. Thus $T \varepsilon \mathfrak{A}''$.

Suppose now that each operator in $\mathfrak{A}$ has a single eigenvalue. Using Engel's theorem, we may assume $\mathfrak{A}$ is in upper-triangular form. Thus each $A \varepsilon \mathfrak{A}$ satisfies

$$R_1 : (Ae_j , e_i) = 0, i > j,$$

$$R_2 : (Ae_i , e_i) - (Ae_j , e_j) = 0.$$

But Lat $\mathfrak{A}^{(2)} \subseteq$ Lat $T^{(2)}$ so T also satisfies R_1 and R_2 .

Remark 4.4. In [6; p. 485] P. Rosenthal raised the question of whether every commutative algebra $\mathfrak{A}$ has the property that $\mathfrak{A} = \mathfrak{A}'' \cap \text{Alg Lat } \mathfrak{A}$. This is known to be the case if $\mathfrak{A}$ is singly generated [4; p. 113, Corollary 1] and is trivially true if $\mathfrak{A}$ is maximal abelian, i.e., $\mathfrak{A} = \mathfrak{A}'$.

Note that by virtue of the last lemma, any algebra satisfying $\mathfrak{A} = \mathfrak{A}'' \cap$ Alg Lat $\mathfrak{A}$ must be 2-reflexive. Thus (Example 4.2) the answer to Rosenthal's question is negative; a counterexample can be found in M_6.

Let $\mathfrak{A}$ be a subalgebra of M_n. A set X of vectors (in $\mathbf{C}_n$) is said to be *separating for* $\mathfrak{A}$ if the only A in $\mathfrak{A}$ annihilating X is the zero operator. I would like to thank Professor Jack McLaughlin for his proof of the following lemma.

LEMMA 4.5. *Let $\mathfrak{A}$ be a commutative subalgebra of M_n each of whose operators has a single eigenvalue. Then either $\mathfrak{A}$ or $\mathfrak{A}^*$ has a separating set of cardinality less than or equal to $n/2$.*

Proof. Note that if X is a separating set for the nilpotent operators in $\mathfrak{A}$, then it is also separating for $\mathfrak{A}$. Thus it suffices to establish the lemma for algebras (without identity) of nilpotents.

Suppose first that $\mathfrak{A}^2 = 0$ and denote by $\eta(\mathfrak{A})$ the null space of $\mathfrak{A}$. If dim $\eta(\mathfrak{A}^*) > n/2$, take X to be a basis for $\eta(\mathfrak{A}^*)^{\perp}$; this will be a separating set for $\mathfrak{A}^*$. On the other hand, if dim $\eta(\mathfrak{A}^*) \leq n/2$, choose X to be a basis for $\eta(\mathfrak{A}^*)$; this will be a separating set for $\mathfrak{A}$ since $\eta(\mathfrak{A}) \supseteq \text{Ran } \mathfrak{A} = \eta(\mathfrak{A}^*)^{\perp}$.

Suppose the lemma were false. Then we could find a minimal counterexample $\mathfrak{A}$. By virtue of the above two paragraphs, $\mathfrak{A}$ will be an algebra of nilpotents and $\mathfrak{A}^2 \neq 0$. Choose an algebra $\mathfrak{B}$ maximal among all subalgebras of $\mathfrak{A}$ whose squares are zero. Without loss of generality, we assume $\mathfrak{B}$ has a separating set X of cardinality less than or equal to $n/2$. (Otherwise, replace $\mathfrak{A}$ by $\mathfrak{A}^*$.) Also note that $\mathfrak{B}$ is an ideal in $\mathfrak{A}$. Choose $A \in \mathfrak{A} \backslash \mathfrak{B}$ such that $AX = 0$. Replacing A by a power if necessary, we can even assume $A^2 \in \mathfrak{B}$. Now for each $B \in \mathfrak{B}$, $AB \in \mathfrak{B}$ and $ABX = 0$. Hence $AB = 0$. Similarly $A^2 = 0$. But this contradicts the maximality of $\mathfrak{B}$ and the lemma is established.

Proof of Theorem 4.1. Let $\mathfrak{A}$ be a commutative subalgebra of M_n and set $\mathfrak{B} = \{B \in M_n \mid \text{Lat } \mathfrak{A}^{(n/2)} \subseteq \text{Lat } B^{(n/2)}\}$. The proof will be complete when we show $\mathfrak{B} \subseteq \mathfrak{A}$.

Case 1. $\mathfrak{A}$ consists of operators with a single eigenvalue.

Applying Lemma 4.3, we see that $\mathfrak{B}$ satisfies the hypothesis of Lemma 4.5. Assume first that $\mathfrak{B}$ has a separating set X of cardinality less than or equal to $n/2$. Given $B \in \mathfrak{B}$, we can find an $A \in \mathfrak{A}$ agreeing with B on X. But then $A - B = 0$ so $\mathfrak{B} \subseteq \mathfrak{A}$.

If $\mathfrak{B}^*$ has the right separating set, the above paragraph gives $\mathfrak{B}^* \subseteq \mathfrak{A}^*$ and hence $\mathfrak{B} \subseteq \mathfrak{A}$.

Case 2. $\mathfrak{A}$ is arbitrary.

Choose a family $\mathcal{J}$ of idempotents in $\mathfrak{A}$ with the following properties.

(1) $\sum \mathcal{J} = I$.

(2) For each $J \in \mathcal{J}$, Ran J is invariant under $\mathfrak{A}$ and each operator in $\mathfrak{A}|_{\text{Ran } J}$ has a single eigenvalue [4; p. 134].

Let $B \in \mathcal{B}$ and fix $J \in \mathcal{J}$. Then BJ belongs to $\mathcal{B}$ and leaves Ran J invariant. Applying Lemma 2.4(2), we see that Lat $(\mathfrak{A}|_{\text{Ran } J})^{(n/2)} \subseteq$ Lat $(BJ|_{\text{Ran } J})^{(n/2)}$. Recalling that $n/2 \geq 2$, it now follows that $BJ \in \mathfrak{A}\, J \subseteq \mathfrak{A}$. (This is a consequence of Case 1 if dim Ran $J \geq 4$, of Theorem 3.1 if dim Ran $J = 3$, and of Corollary 2.6 otherwise). Thus $B = \sum_J BJ$ belongs to $\mathfrak{A}$, and the proof is complete.

5. An extension of Theorem 3.1.

Example 5.1. It may happen that $\mathcal{H}$ can be expressed as the direct sum $\mathcal{H} = M_1 \oplus M_2$ of subspaces invariant under $\mathfrak{A}$ such that $\mathfrak{A}|_{M_1}$, $\mathfrak{A}|_{M_2}$ are k-reflexive but $\mathfrak{A}$ is not k-reflexive. Indeed take $\mathfrak{A} = \{A \in M_m \oplus M_m \mid A$ is upper-triangular, A has a single eigenvalue, and the supertrace of A is zero$\}$. Then it is easily seen that $\mathfrak{A}|_{\mathbf{C}_m \oplus (0)}$ and $\mathfrak{A}|_{(0) \oplus \mathbf{C}_m}$ have characterizations of order 2 and hence are 2-reflexive. On the other hand, an argument similar to that of Example 3.2 shows $\mathfrak{A}$ is not $(m - 2)$-reflexive. The following extension of Theorem 3.1 may therefore be somewhat surprising.

THEOREM 5.2. *Let k be an integer greater than or equal to 2 and $\mathfrak{A}$ a subalgebra of $\mathcal{L}(\mathcal{H})$. Suppose $\mathcal{H}$ can be expressed as the direct sum of subspaces in* Lat $\mathfrak{A}$ *each of which has dimension $k + 1$ or less. Then $\mathfrak{A}$ is k-reflexive.*

COROLLARY 5.3. *Suppose $n \geq 3$. Then every subalgebra $\mathfrak{A}$ of $M_n \oplus \cdots \oplus M_n$ is $(n - 1)$-reflexive.*

Proof. The subspaces $\mathbf{C}_n \oplus 0 \oplus \cdots \oplus 0, \cdots, 0 \oplus \cdots \oplus 0 \oplus \mathbf{C}_n$ are invariant for $\mathfrak{A}$.

The following two lemmas are essentially established in the proofs of Lemma 3.4 and Theorem 3.1 respectively.

LEMMA 5.4. *Let φ be a linear functional on $\mathcal{L}(\mathcal{H})$ defined by $\varphi(A) = \sum_{i=1}^{n} (Ax_i, y_i)$, $x_i, y_i \in \mathcal{H}$. Suppose rank $\langle x_1 \cdots x_n \rangle \leq k$, $k \neq 0$. Then there exist vectors x and y in $\mathcal{H}^{(k)}$ such that $\varphi(A) = (A^{(k)}x, y)$ for all $A \in \mathcal{L}(\mathcal{H})$.*

LEMMA 5.5. *Let $\mathcal{B}$ be a subalgebra of $\mathcal{L}(\mathcal{H})$ such that each operator in $\mathcal{B}$ has a single eigenvalue. Suppose moreover that $\mathcal{H}$ has dimension $(k + 1) \geq 3$ and φ is a linear functional on $\mathcal{L}(\mathcal{H})$. Then there exist vectors x and y in $\mathcal{H}^{(k)}$ such that $\varphi(B) = (B^{(k)}x, y)$ for all $B \in \mathcal{B}$.*

When the algebra $\mathfrak{A}$ is commutative, the idempotents constructed in the following lemma correspond to the "Jordan blocks" of [4; p. 134]. Lemma 5.6 may therefore be regarded as a noncommutative analogue of the theorem in [4].

LEMMA 5.6. *Let $\mathfrak{A}$ be a subalgebra of $\mathcal{L}(\mathcal{H})$. Then there exists a family $\mathcal{J}$ of commuting idempotents in $\mathfrak{A}$ satisfying the following.*

(1) *$\mathcal{J}$ is supplementary.*

(2) *Suppose $J \in \mathcal{J}$ and $M \in$ Lat $\mathfrak{A}$ is a subset of* Ran J. *Then $\mathfrak{A}|_M$ consists of operators with a single eigenvalue.*

Proof. Find a maximal (with respect to inclusion) commuting family $\mathfrak{F}$ of idempotents in $\mathfrak{A}$. Suppose M belongs to Lat $\mathfrak{A}$ and $\mathfrak{A}|_M$ contains operators with more than one eigenvalue.

Claim. There is an $F_0 \in \mathfrak{F}$ such that $F_0|_M$ is a proper idempotent, i.e., $F_0|_M$ is neither 0 nor the identity; this is the same as saying $F_0(M)$ is a proper subset of M. Since $\mathfrak{A}|_M$ contains operators with more than one eigenvalue, we can find an operator $R \in \mathfrak{A}$ such that $R|_M$ is a proper idempotent. Suppose the claim failed and set $F_1 = \pi\{F \in \mathfrak{F} \mid F|_M = I\}$. Then for each $F \in \mathfrak{F}$ either F or $I - F$ must appear as a factor of F_1. Thus taking $F_0 = F_1RF_1$, we see that F_0 commutes with each operator in $\mathfrak{F}$ and $F_0|_M$ is still a proper idempotent. Replacing F_0 by an appropriate polynomial in F_0, we can even assume F_0 is an idempotent. But this contradicts the maximality of $\mathfrak{F}$ and the claim is established.

Take $\mathcal{J}$ to be the supplementary subset of $\mathfrak{F}$ of maximal cardinality. Clearly $\mathcal{J}$ satisfies (1). To see that it also satisfies (2), suppose $J \in \mathcal{J}$ and $M \in$ Lat $\mathfrak{A}$ is a subset of Ran J. Then $J|_M$ would be the identity. By our claim and the maximality of $\mathcal{J}$, we conclude that $\mathfrak{A}|_M$ consists of operators with a single eigenvalue and the lemma is established.

Proof of Theorem 5.2. Find a collection $\mathfrak{M} \subseteq$ Lat $\mathfrak{A}$ such that (1) $\mathcal{H}$ is the (vector space) direct sum of the elements of $\mathfrak{M}$ and (2) each $M \in \mathfrak{M}$ has dimension $k + 1$ or less. By redefining the inner product on $\mathcal{H}$, we can even assume the subspaces in $\mathfrak{M}$ are orthogonal to each other. For $M \in \mathfrak{M}$ denote by P_M the orthogonal projection with range M. Note that each P_M commutes with $\mathfrak{A}$ and $\sum_M P_M = I$.

Let $\mathcal{C} = \{$relations of order less than or equal to k satisfied by $\mathfrak{A}\}$ and set $\mathcal{B} = \{B \in \mathcal{L}(\mathcal{H}) \mid B$ satisfies $\mathcal{C}\}$. Suppose φ is an arbitrary linear functional on $\mathcal{L}(\mathcal{H})$ which annihilates $\mathfrak{A}$.

Claim. φ annihilates $\mathcal{B}$. This will complete the proof for then $\mathfrak{A}$ will equal $\mathcal{B}$, i.e., $\mathcal{C}$ will be a characterization of order k for $\mathfrak{A}$.

Write φ in the form $\varphi(B) = \sum_{i=1}^n (Bx_i, y_i)$ and choose $\mathcal{J}$ for $\mathfrak{A}$ as in Lemma 5.6. Note that by virtue of Lemma 4.3 the same $\mathcal{J}$ works for $\mathcal{B}$ and each P_M commutes with $\mathcal{B}$.

Case 1. For some fixed $J_0 \in \mathcal{J}$, $x_1 \cdots x_n$ all belong to Ran J_0. For each $M \in \mathfrak{M}$ we define a linear functional φ_M on $\mathcal{L}(\mathcal{H})$ by $\varphi_M(B) = \varphi(BP_M)$. Then we have $\varphi_M(B) = \sum_{i=1}^n (BP_Mx_i, P_My_i)$ for all $B \in \mathcal{B}$.

Note that $\{P_Mx_i\}_{i=1}^n$ all belong to $J_0(M)$ and rank $\langle P_Mx_1, \cdots, P_Mx_n\rangle$ is less than or equal to dim $M \leq k + 1$. Moreover, if rank $\langle P_Mx_1, \cdots, P_Mx_n\rangle = k + 1$, then $M \subseteq$ Ran J_0 and Lemma 5.6 implies that $\mathcal{B}|_M$ consists of operators with a single eigenvalue. Thus, viewing φ_M as a linear functional on $\mathcal{B}|_M$ we conclude that either Lemma 5.4 or Lemma 5.5 is applicable. Hence, we get vectors x_M, y_M in $M^{(k)}$ such that $\varphi_M(B) = (B^{(k)}x_M, y_M)$ for all $B \in \mathcal{B}$.

Set $x = \sum_M x_M$ and $y = \sum_M y_M$. Then $\varphi(B) = (B^{(k)}x, y)$ for all $B \in \mathcal{B}$. Recalling the fact that φ annihilates $\mathfrak{A}$ and the definition of $\mathcal{B}$, we see that φ annihilates $\mathcal{B}$ as desired.

Case 2. $x_1, \cdots, x_n$ are arbitrary. Fix $J \in \mathcal{J}$ and define ψ_J by $\psi_J(B) = \varphi(BJ)$. Since $J \in \mathfrak{A}$, we see that ψ_J annihilates $\mathfrak{A}$. Moreover, Case 1 applies to each ψ_J so each ψ_J annihilates $\mathcal{B}$. It follows that $\varphi = \sum_{J \in \mathcal{J}} \psi_J$ also annihilates $\mathcal{B}$.

Remark. Note that in the preceding proof the idempotents in $\mathcal{J}$ belong to $\mathfrak{A}$ while the projections $\{P_M\}$ only belong to $\mathfrak{A}'$. This is why the linear functionals $\{\varphi_M\}$ are more difficult to handle than the $\{\psi_J\}$.

We conclude this section with a commutative analogue of Theorem 5.2. The proof requires the following corollary of Lemma 4.5.

LEMMA 5.7. *Let $\mathcal{B}$ be a commutative subalgebra of $\mathcal{L}(\mathcal{H})$ such that each operator in $\mathcal{B}$ has a single eigenvalue. Suppose moreover that $\mathcal{H}$ has dimension less than or equal to $2k + 1$, $k \geq 2$, and that φ is a linear functional on $\mathcal{L}(\mathcal{H})$. Then there exist vectors x and y in $\mathcal{H}^{(k)}$ such that $\varphi(B) = (B^{(k)}x, y)$ for all $B \in \mathcal{B}$.*

Proof. Assume first that $\mathcal{B}$ has a separating set $\{x_1, \cdots, x_k\}$. Set $x = x_1 \oplus \cdots \oplus x_k$ and define a linear functional $\tilde{\varphi} : \mathcal{B}^{(k)}x \rightarrow \mathbf{C}$ by $\tilde{\varphi}(B^{(k)}x) = \varphi(B)$, $B \in \mathcal{B}$. Then $\tilde{\varphi}$ is well-defined and hence there is a vector $y \in \mathcal{B}^{(k)}x$ such that $\varphi(B) = \tilde{\varphi}(B^{(k)}x) = (B^{(k)}x, y)$ for all $B \in \mathcal{B}$.

If $\mathcal{B}^*$ has the proper separating set, apply the argument of the preceding paragraph to the linear functional φ^* defined (on $\mathcal{B}^*$) by $\varphi^*(B^*) = \overline{\varphi(\mathrm{B})}$, $B^* \in \mathcal{B}^*$.

THEOREM 5.8. *Let k be an integer greater than or equal to 2 and $\mathfrak{A}$ a commutative subalgebra of $\mathcal{L}(\mathcal{H})$. Suppose $\mathcal{H}$ can be expressed as the direct sum of subspaces in* Lat $\mathfrak{A}$, *each of which has dimension $2k + 1$ or less. Then $\mathfrak{A}$ is k-reflexive.*

Proof. Find a collection $\mathcal{M} \subseteq \text{Lat } \mathfrak{A}$ such that (1) $\mathcal{H}$ is the direct sum of the elements in $\mathcal{M}$, (2) each $M \in \mathcal{M}$ has dimension $2k + 1$ or less and (3) $\mathfrak{A}|_M$ consists of operators with a single eigenvalue.

The proof now proceeds exactly as that of Theorem 5.2 with the following differences.

(a) Take $\mathcal{J} = \{I\}$. This of course obviates the need for Case 2.

(b) We have rank $\langle P_M x_1 \cdots P_M x_n \rangle \leq 2k + 1$ and $\mathcal{B}|_M$ consists of operators with a single eigenvalue. Thus, we appeal to Lemma 5.7 instead of Lemma 5.4 or Lemma 5.5.

6. Concluding remarks.

Remark 6.1. In order for a set of operators to be k-reflexive for some k, it must be an identity containing algebra. This is because all subspaces are invariant under I.

Remark 6.2. It is a well-established technique to study the structure of von Neumann algebras (possibly acting on an infinite dimensional space) via their linear functionals. The point of the present paper is that on finite dimensional spaces, this technique can be used to study non-self-adjoint algebras.

Remark 6.3. Let V be a finite dimensional vector space over a field F and write V^* for the vector space of linear functionals from V to F. Of course, it makes sense to speak of k-reflexive subalgebras of $\mathcal{L}(V)$, and by replacing the inner product structure on Hilbert spaces by the duality between V and V^*, the various definitions and theorems of this paper can be interpreted in this general context. (For example, we write for Proposition 2.4(3), for each $x_1 \cdots x_k \in V$ and $y_1 \cdots y_k \in V^*$, $\cdots$.) Moreover the proofs of Section 2 carry over in this setting.

Thus the inner product structure on $\mathbf{C}$ plays an innocuous role in the considerations of this paper. Beginning with the proof of Theorem 3.6 however, we have made use of one property of the complex numbers: every matrix over $\mathbf{C}$ has an upper-triangular Jordan canonical form. The latter, of course, is valid for any field F which is algebraically closed and it is a more or less routine matter to check that the entire paper remains valid in this context.

What of fields F which are not algebraically closed? The author knows of no counterexample to the generalized version of Theorem 3.1 and suspects that it may indeed be true. What is needed for the proof is some sort of generalized Engel's theorem of the following type.

Conjecture. Suppose $\mathfrak{A}$ is a subalgebra of $M_n(F)$ having no proper idempotents. Then the operators in $\mathfrak{A}$ can be simultaneously put in "pseudo-Jordan form."

These matters will be left for future investigations.

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