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A Dichotomy for Linear Spaces of Toeplitz Operators*

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Let **S** be a linear manifold of bounded Hilbert space operators. An operator A belongs to the *reflexive closure* of **S** if Af belongs to the closure of $\mathbf{S}f$ for each vector f in the underlying Hilbert space. Two extreme possibilities are (1) **S** is *reflexive* in the sense that $\operatorname{ref}\mathbf{S} = \mathbf{S}$, and (2) **S** is *transitive* in the sense that $\operatorname{ref}\mathbf{S}$ includes all bounded operators on the underlying space. We show that every linear space $\mathscr{B}$ of Toeplitz operators which is closed in the ultraweak operator topology is either transitive or reflexive. No intermediate behavior is possible. The full space of all Toeplitz operators is transitive, but if $\mathscr{B}$ is properly contained in this space and contains all *analytic* Toeplitz operators, then $\mathscr{B}$ must be reflexive. In particular, the space of Toeplitz operators whose matrices have zeros on a fixed superdiagonal is reflexive.

1. INTRODUCTION

In [Sar], D. Sarason proved that the algebra $\mathscr{A}$ of *analytic* Toeplitz operators is reflexive. Moreover, it is elementary in the sense that every ultraweakly continuous linear functional on $\mathscr{A}$ is induced by an operator of rank at most one. It follows that $\mathscr{A}$ is *hereditarily reflexive*, i.e., that every ultraweakly closed subspace of $\mathscr{A}$ is also reflexive.

In this paper, we study the full space $\mathscr{T}$ of all Toeplitz operators and its various ultraweakly closed subspaces. It is fairly easy to see that $\mathscr{T}$ is transitive

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* A preliminary version of this paper, under the title "Reflexive closures of spaces of Toeplitz operators," was presented at the 1996 AMS annual meeting in Orlando; the main results in the current paper resolve conjectures raised in that talk.

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and non-elementary. The main result of the paper is the dichotomy referred to in the title; a similar dichotomy (under a different notion of reflexivity) occurs in the setting of AF algebras studied by J. Peters and W. Wogen [PW].

THEOREM 1.1. *Every intransitive, ultraweakly closed subspace of $\mathscr{F}$ is reflexive.*

While it is easy to construct transitive subspaces of $\mathscr{F}$—such spaces can have infinite codimension, these are incomparable to $\mathscr{A}$.

THEOREM 1.2. *Every proper, ultraweakly closed subspace of $\mathscr{F}$ which contains $\mathscr{A}$ is reflexive.*

Both of these theorems generalize Sarason's result. Theorem 1.1 has some surprising consequences. For example, in general, there is no reason to expect reflexive spaces to be hereditarily reflexive—after all, $\mathbf{B}(\mathbf{H})$ is reflexive, but has many non-reflexive subspaces. It is a consequence of Theorem 1.1 that this pathology cannot occur inside $\mathscr{F}$. Indeed, since it is clear that transitivity is "ancestral", 1.1 implies that reflexivity is hereditary.

PROPOSITION 1.3. *Every reflexive subspace of $\mathscr{F}$ is hereditarily reflexive.*

This might suggest that that only "small" subspaces of $\mathscr{F}$ have a chance of being reflexive. On the other hand, it is easy to construct intransitive hyperplanes in $\mathscr{F}$, e.g., the set of Toeplitz operators whose matrices relative to standard basis for H^2 have zeros on a fixed superdiagonal. Thus the following consequence of 1.1 implies that $\mathscr{F}$ has many reflexive hyperplanes.

PROPOSITION 1.4. *Every ultraweakly closed, intransitive hyperplane in $\mathscr{F}$ is reflexive.*

Conversely, it is clear that 1.3 and 1.4 together imply 1.1. This is the route we will take in constructing our proofs. Tools used in the paper include the interplay between function theory and invariant subspace theory which goes back to A. Beurling, supplemented by the use of duality highlighted in S. Brown's treatment of subnormal operators. Pleasant properties of the symbol map between $L^\infty(\mathbb{T})$ and $\mathscr{F}$ reduce the study of linear functionals on (subspaces of) $\mathscr{F}$ to function theoretic questions. In particular, J. Bourgain's [Bo] characterization of those members of $L^1(\mathbb{T})$ which can be expressed in the form $g\bar{h}$ with g, h belonging to the Hardy space H^2 leads to the following version of Theorem 1.1.

THEOREM 1.1′. *Suppose $\mathscr{B}$ is a linear space of Toeplitz operators which is closed in the ultraweak operator topology. Then the following are equivalent.*

(1) *$\mathscr{B}$ is intransitive.*

(2) *There is an integrable function f on the unit circle such that* $\log |f|$ *is also integrable and* $\int f\phi = 0$ *whenever ϕ is the symbol of a member of $\mathscr{B}$.*

(3) *$\mathscr{B}$ is reflexive.*

Moreover, Bourgain's result implies that $H^2\overline{H^2}$ "just misses" exhausting L^1 and this plays a crucial role in the proof of 1.3. The proof of 1.4, on the other hand, does not depend on Bourgain's characterization. Instead, it exploits the fact that $g\bar{h} = x\bar{y}$ for many different pairs x, $y \in H^2$.

We would like to thank Hari Bercovici for bringing Bourgain's result to our attention and Graeme West for expository suggestions concerning Section 2.

Now let us recall some basic notations and definitions, which will be used in the paper. For a Hilbert space $\mathbf{H}$ we will write $\mathbf{B}(\mathbf{H})$ for the algebra of all bounded linear operators and $\mathbf{T}$ (or $\mathbf{T}(\mathbf{H})$ when the underlying Hilbert space must be specified) for the ideal of trace class operators in $\mathbf{B}(\mathbf{H})$, regarded as equipped with the trace norm $\|\cdot\|_1$. We also denote the ideal of finite rank operators in $\mathbf{B}(\mathbf{H})$ by $\mathbf{F}$ and set $\mathbf{F}_k \equiv \{T \in \mathbf{F}\colon \text{rank } T \leqslant k\}$. Members of $\mathbf{F}_1$ take the special form

$$z \mapsto (x \otimes y)z \equiv (z, y)x, \qquad x, y, z \in \mathbf{H}.$$

We follow the usual practice of using the bilinear form

$$\langle A, T\rangle \equiv \text{trace}\,(AT), \qquad A \in \mathbf{B}(\mathbf{H}), \quad T \in \mathbf{T} \tag{1.1}$$

to regard members of $\mathbf{T}$ as linear functionals on $\mathbf{B}(\mathbf{H})$ and vice verse. The crucial fact is that $\mathbf{B}(\mathbf{H})$ is the Banach space dual of $\mathbf{T}$. The resulting weak-star topology on $\mathbf{B}(\mathbf{H})$ coincides with the so called ultraweak operator topology; moreover $\sigma(\mathbf{B}(\mathbf{H}), \mathbf{F})$ is the weak operator topology. The reader is referred to [Sch] or [Di] for background on duality.

The reader should perhaps be warned that we are following the traditional, but slightly inconsistent, use of inner product notation. For f, $g \in L^2$ (square-integrable functions with respect to an understood measure), we set $(f, g) \equiv \int f\bar{g}$ in good Hilbert-space fashion, but there is no corresponding adjunction of T in Display 1.1.

The following definition, originally due to V. S. Shulman, is a basic one for the paper.

Definition 1.5. The *reflexive closure* of a linear manifold $\mathbf{S}$ in $\mathbf{B}(\mathbf{H})$ is given by

$$\text{ref}\,\mathbf{S} = \{B \in \mathbf{B}(\mathbf{H})\colon Bf \in \overline{\mathbf{S}f} \text{ for all } f \in \mathbf{H}\}.$$

$\mathbf{S}$ is said to be *reflexive* if $\text{ref}\,\mathbf{S} = \mathbf{S}$ and *transitive* if $\text{ref}\,\mathbf{S} = \mathbf{B}(\mathbf{H})$.

When $\mathscr{A}$ is an identity-containing operator algebra, we have ref $\mathscr{A} =$ alglat $\mathscr{A}$, but this will play a minimal role below.

Our main tool for computing reflexive closures in this paper is the observation that an operator $B \in \mathbf{B}(\mathbf{H})$ fails to belong to ref $\mathbf{S}$ if and only if there are non-zero vectors $f, g \in \mathbf{H}$ satisfying $(Af, g) = 0$ for all $A \in \mathbf{S}$, but $(Bf, g) \neq 0$, i.e., if and only if there is a rank one operator $f \otimes g$ which separates B from $\mathbf{S}$. Thus the linear manifold $\mathbf{S}$ is reflexive if and only if each operator in $\mathbf{B}(\mathbf{H}) \backslash \mathbf{S}$ can be separated from $\mathbf{S}$ by some member of $\mathbf{F}_1$. In particular, for ultraweakly closed $\mathbf{S}$, reflexivity is tantamount to having $\mathbf{S}_\perp \cap \mathbf{F}_1$ total in $\mathbf{S}_\perp$.

Emphasis on the relationship between rank one operators and invariant subspaces is largely attributable to Scott Brown's work, which also focused attention on the following concept, originally due to A. I. Loginov and V. S. Shulman [LS].

Definition 1.6. A linear manifold $\mathbf{S}$ in $\mathbf{B}(\mathbf{H})$ is *elementary* if $\mathbf{S}_\perp + \mathbf{F}_1 = \mathbf{T}$.

Surveys of the early work on reflexivity include H. Radjavi and P. Rosenthal's book [RR] and the article [NLS] by M. A. Naimark, A. I. Loginov, and V. S. Shulman. Except for changes in font, we are following the terminology and notation of [Az]; the more general concepts obtained by replacing $\mathbf{F}_1$ by $\mathbf{F}_k$ in the preceding discussion are also treated there. Namely, a linear manifold $\mathbf{S}$ is k-elementary if $\mathbf{S}_\perp + \mathbf{F}_k = \mathbf{T}$; it is k-transitive if $\mathbf{S}_\perp \cap \mathbf{F}_k = \{0\}$. Moreover, an ultraweakly closed linear manifold $\mathbf{S}$ is k-reflexive if $\mathbf{S}_\perp \cap \mathbf{F}_k$ is total in $\mathbf{S}_\perp$.

The following fact, also originally due to Loginov and Shulman, effectively reduces Proposition 1.3 to a function–theory question. We include a proof here for the convenience of the reader.

Proposition 1.7. *A reflexive space is hereditarily reflexive if and only if it is elementary.*

Proof. Let $\mathbf{S}$ be reflexive subspace of $\mathbf{B}(\mathbf{H})$. Suppose first that $\mathbf{S}$ is hereditarily reflexive and $t \in \mathbf{T}$. If $t \in \mathbf{S}_\perp$ then $t \in \mathbf{S}_\perp + \mathbf{F}_1$. If $t \notin \mathbf{S}_\perp$ then $t^\perp \cap \mathbf{S}$ is a hyperplane in $\mathbf{S}$. Choosing $b \in \mathbf{S} \backslash t^\perp$, we apply reflexivity of $t^\perp \cap \mathbf{S}$ to find $f \in \mathbf{F}_1$ which separates this hyperplane from b. Normalizing so that $\langle b, f \rangle = \langle b, t \rangle$, we see that $t - f \in \mathbf{S}_\perp$ and $t \in \mathbf{S}_\perp + \mathbf{F}_1$. Hence $\mathbf{S}$ is elementary.

Suppose conversely that $\mathbf{S}$ is elementary, that $\mathbf{M}$ is one of its proper ultraweakly closed subspaces, and that $b \in \mathbf{B}(\mathbf{H}) \backslash \mathbf{M}$. To see that $\mathbf{M}$ is reflexive, we must find an $f \in \mathbf{F}_1$ which separates b from $\mathbf{M}$. If $b \notin \mathbf{S}$, this follows by the reflexivity of $\mathbf{S}$. On the other hand, if $b \in \mathbf{S} \backslash \mathbf{M}$, then ultra-weak closure of $\mathbf{M}$ at least provides a $t \in \mathbf{T}$ which does the separation. Appealing to the hypothesis that $\mathbf{S}$ is elementary, we find an $f \in \mathbf{F}_1$ which agrees with t on $\mathbf{S}$ and hence also serves to separate b from $\mathbf{M}$.

The structure of the balance of the paper is as follows. After reviewing some properties of the symbol map in the next section, we first investigate the properties of the space $\mathscr{F}$ of all Toeplitz operators in Section 3 and then apply Bourgain's result to prove Proposition 1.3. Most of Section 4 is devoted to establishing Proposition 1.4, after which the proofs of Theorems 1.1 and 1.2 are easily completed. The final section presents some applications of the main results and raises the question of metric versions of 1.1–1.4.

2. TOEPLITZ OPERATORS AND LINEAR FUNCTIONALS

We write $\mathbb{D}$ for the open unit disc in the complex plane $\mathbb{C}$ and $\mathbb{T}$ for its boundary; we equip this unit circle $\mathbb{T}$ with normalized Lebesgue measure, denoted m in what follows. For $1 \leqslant p \leqslant \infty$, we write L^p for the Banach space $L^p(\mathbb{T}, m)$ and H^p for the corresponding Hardy space. We reserve the letter P for the projection operator from L^2 onto H^2 and S for the unilateral shift, realized as $(Sf)(z) = zf(z)$ for $f \in H^2$.

Definition 2.1. For each $\phi \in L^\infty$, the operator $T_\phi \colon H^2 \to H^2$ is defined by

$$T_\phi f = P(\phi f), \qquad \text{for} \quad f \in H^2.$$

This is referred to as the *Toeplitz operator T_ϕ with symbol ϕ*. The *symbol map* is the function $\xi \colon L^\infty \to \mathbf{B}(H^2)$ defined by $\xi(\phi) = T_\phi$.

The reader is referred to [Ho], [Ha], and [Do] for background on these concepts.

Proposition 2.2. *The symbol map has the following properties.*

(1) *ξ is a linear isometry from L^∞ into $\mathbf{B}(H^2)$*

(2) *$\xi(L^\infty)$ consists of those operators $A \in \mathbf{B}(H^2)$ satisfying $S^*AS = A$.*

(3) *$\xi(H^\infty)$ consists of those operators $A \in \mathbf{B}(H^2)$ satisfying $AS = SA$.*

Proof. These can all be found in the last chapter of [Ha]: (1) is Corollary 1 to Problem 196, (2) is Corollary 1 to Problem 194, and (3) is Problem 116.

Thus ξ should be thought of as a non-multiplicative functional calculus. We will often exploit the fact [Do] that it is partially multiplicative in the sense that $T_{gh}=T_g T_h$ whenever $g\in\overline{H^2}$ or $h\in H^2$. The interface between duality and H^∞-functional calculi has come to play an important role in invariant subspace theory. The reader should consult [BFP] for a good account of these developments. The following corollary adapts some of these ideas to the current setting.

Corollary 2.3. *Set* $\mathscr{F} := \xi(L^\infty)$ *and* $\mathscr{A} := \xi(H^\infty)$.

(1) L^∞ *and* $\mathscr{F}$ *each have unique weak-star topologies.*

(2) ξ *provides a homeomorphism between* L^∞ *and* $\mathscr{F}$ *when they are equipped with their weak-star topologies.*

(3) $\mathscr{F}$ *and* $\mathscr{A}$ *are closed in the weak operator topology.*

(4) *The weak-star topology on* $\mathscr{F}$ *coincides with the relative ultraweak operator topology it inherits as a subspace of* $\mathbf{B}(H^2)$.

Proof. Since L^∞ is a von Neumann algebra, it has a unique predual, [Sak, Section 1.13], i.e., the canonical images of all such preduals in $(L^\infty)^*$ coincide. Thus (1) and (2) follow from the first part of Proposition 2.2. Since the sets of operators satisfying the equations in the remaining parts of that proposition are closed in the weak operator topology, we also have (3) of the current result. A fortiori, $\mathscr{F}$ is a $\sigma(\mathbf{B}(H^2), \mathbf{T})$-closed operator space, whence the quotient $\mathbf{T}/(\mathscr{F}_\perp)$ also provides a predual of $\mathscr{F}$ and (4) is a consequence of (1).

Remark 2.4 T. Ando [An] showed that H^∞ also has a unique predual. Z. Ruan [R] has recently investigated the question of which weak-star closed operator algebras have unique preduals.

As a consequence of Corollary 2.3 we can characterize linear functionals on $\mathscr{F}$.

Corollary 2.5. *Each* $f\in L^1$ *induces a linear functional* $\hat{f}$ *on* $\mathscr{F}$ *by*

$$\hat{f}(T_{\phi)}=\int \phi f\,dm, \qquad \phi\in L^\infty.$$

These linear functionals are ultraweakly continuous on $\mathscr{F}$ *and every ultraweakly continuous linear functional on* $\mathscr{F}$ *takes this form.*

Proof. We follow the usual practice of identifying the unique predual of L^∞ with L^1. Thus the preceding corollary tells us that $(\xi^*)^{-1}$ maps L^1

onto the unique predual of $\mathscr{T}$. The proof is completed by the following computation, valid for all $f \in L^1$ and $\phi \in L^\infty$:

$$\langle T_\phi, (\xi^*)^{-1} f \rangle = \langle \xi^{-1}(T_\phi) f \rangle = \langle \phi, f \rangle.$$

3. THE FULL SPACE OF TOEPLITZ OPERATORS $\mathscr{T}$

In terms of preannihilators, the following theorem means that while $\mathscr{T}_\perp$ has no rank one members, it is nevertheless rich in rank two members. The latter assertion is easy to understand matricially. Indeed, relative to the standard orthonormal basis $\{z^{n-1}\}_{n=1}^{\infty}$ for H^2, members of $\mathscr{T}$ correspond to matrices which are constant on diagonals, so that members of $\mathscr{T}_\perp$ correspond to matrices each of whose diagonals sum to zero; matrices with precisely two non-zero entries form a total set in this space.

Theorem 3.1. *$\mathscr{T}$ is transitive but 2-reflexive.*

Proof. Suppose the rank one operator $f \otimes g$ belongs to $\mathscr{T}_\perp$. Then for any $\phi \in L^\infty$ we have

$$0 = \langle T_\phi, f \otimes g \rangle = (T_\phi f, g) = \int \phi f \bar{g}\, dm.$$

Hence $f\bar{g} \equiv 0$. Since $f, g \in H^2$, the corollary on page 52 of [Ho] tells us that $f \equiv 0$ or $g \equiv 0$. Hence $\mathscr{T}_\perp \cap \mathbf{F}_1 = \{0\}$ and $\mathscr{T}$ is transitive.

For the second part of the theorem, suppose $B \notin \mathscr{T}$. Apply the second part of Proposition 2.2 to get $f, g \in H^2$ with $((S^*BS - B) f, g) \neq 0$. This means the rank two operator $Sf \otimes Sg - f \otimes g$ separates B from $\mathscr{T}$, whence $\mathscr{T}_\perp \cap \mathbf{F}_2$ is total in $\mathscr{T}_\perp$.

The following result of J. Bourgain [Bo] will play a key role in the sequel.

Proposition 3.2. *A non-zero function $f \in L^1$ belongs to $H^2\overline{H^2}$ if and only if* $\log |f| \in L^1$.

Corollary 3.3. *H^1 is a proper subset of $H^2\overline{H^2}$ which is in turn a proper subset of L^1. However, $L^1 = H^2\overline{H^2} + \mathbb{C}g$ for each g in $H^2\overline{H^2}$ other than the zero function.*

Proof. The first inclusion follows from 3.2 since $\log |f| \in L^1$ for every $f \in H^1$; containment is proper since all real-valued members of H^1 are constant. The second inclusion is proper since any member of H^2 which vanishes on a set of positive measure must vanish identically.

We use a slight modification of the argument in the second paragraph on page 48 of [Bo] to establish the final assertion. Fix $g \in H^2\overline{H^2}\setminus\{0\}$ and let $f \in L^1$ be given. Define a function p on almost all of $\mathbb{T}\times\mathbb{T}$ by $p(z, w) = \log|f(z)+wg(z)|$. Since $\log x \leqslant x$ for $x \geqslant 1$, we always have $p^+(z, w) \leqslant |f(z)| + |g(z)|$. On the other hand, it follows from Jensen's formula, [Ho, p. 68] that

$$\int_{\mathbb{T}} p(z, w)\, dm(w) = \max\{\log|f(z)|, \log|g(z)|\}$$

Since f, g, $\log|g| \in L^1$ by hypothesis, the last two observations combine to tell us that $p \in L^1(\mathbb{T}\times\mathbb{T})$. Applying Tonelli's Theorem, we conclude that $\int_{\mathbb{T}} p(z, w)\, dm(z)$ is finite for almost all w, whence $f + wg \in H^2\overline{H^2}$ for almost all w as well.

The following theorem implies that every reflexive subspace of $\mathscr{F}$ is hereditarily reflexive.

THEOREM 3.4. *$\mathscr{F}$ is not elementary, but every intransitive subspace of $\mathscr{F}$ is elementary.*

Proof. Take E to be a subset of $\mathbb{T}$ having measure strictly between 0 and 1, and write f for its characteristic function. Suppose that $\hat{f}$ agreed with some $h \otimes k \in \mathbf{F}_1$ on $\mathscr{F}$. Then we would have $\int \phi(f - h\bar{k}) = 0$ for each $\phi \in L^\infty$ whence $h\bar{k}$ would vanish on a set of positive measure. This, in turn, would force $h \equiv 0$ or $k \equiv 0$. This contradiction shows $\hat{f} \notin \mathscr{F}_\perp + \mathbf{F}_1$ and hence that $\mathscr{F}$ is not elementary.

Suppose now that $\mathscr{B}$ is a intransitive subspace of $\mathscr{F}$ and fix a non-trivial rank-one operator $g_1 \otimes h_1$ in its preannihilator. Now let θ be an ultraweakly continuous linear functional. Apply Corollary 2.5 to find $f \in L^1$ with $\theta(T_\phi) = \int \phi f\, dm$ for each $\phi \in L^\infty$. Next apply 3.3 to get $g_2, h_2 \in H^2$ satisfying $f = g_1\hat{h}_1 + g_2\hat{h}_2$. This means $\theta = \hat{f}$ agrees with $g_1 \otimes h_1 + g_2 \otimes h_2$ on $\mathscr{F}$ whence θ agrees with $g_2 \otimes h_2$ on $\mathscr{B}$ and the proof is complete.

Proof of Proposition 1.3 The last result tells us that every reflexive subspace of $\mathscr{F}$ is elementary, so we need only appeal to Proposition 1.7.

While proper subspaces of $\mathscr{F}$ can be as badly behaved as $\mathscr{F}$ itself, spaces which contain $\mathscr{A}$ are more tractable; the following corollary points out this contrast.

COROLLARY 3.5. *Suppose $\mathscr{B}$ is a proper ultraweakly closed subspace of $\mathscr{F}$.*

(1) *$\mathscr{B}$ is 2-elementary and 2-reflexive.*

(2) *In order for $\mathscr{B}$ to be reflexive, it is necessary and sufficient that* ref $\mathscr{B}$ *be contained in $\mathscr{T}$.*

(3) *It is possible for $\mathscr{B}$ to be transitive, non–elementary, and have infinite codimension in $\mathscr{T}$.*

(4) *On the other hand, $\mathscr{B}$ cannot be transitive if it contains $\mathscr{A}$.*

Proof. The last paragraph of the proof of Theorem 3.4 shows that $\mathscr{T}$ is 2-elementary and hence so are all of its ultraweakly closed subspaces. Since we already know $\mathscr{T}$ is 2-reflexive, all such subspaces are 2-reflexive by 2.8 and 2.10 of [Az].

As for (2), the condition is obviously necessary. On the other hand, since ref $\mathscr{B}$ is clearly reflexive, knowing that ref $\mathscr{B} \subset \mathscr{T}$ allows us to apply Theorem 3.4 to conclude that ref $\mathscr{B}$ is *hereditarily* reflexive.

To construct an example for (3), fix disjoint subsets E, F of $\mathbb{T}$ each having measure $\frac{1}{3}$. Take $\mathscr{B} = \{T_\phi \colon \phi = 0 \text{ on } E\}$, the set of Toeplitz operators whose symbols vanish on E. Clearly, $\mathscr{B}$ is ultraweakly closed. Suppose the operator $g \otimes h$ belonged to $\mathscr{B}_\perp$. Then $g\bar{h}$ would have to be orthogonal to every L^∞ function which vanishes on E, so $g\bar{h}$ must be supported on E. Since $g, h \in H^2$, it follows that $g\bar{h} \equiv 0$ and we have shown that there are no rank one operators in $\mathscr{B}_\perp$.

Next write χ_F for the characteristic function of F. Suppose that $\hat{\chi}_F$ agreed with some $g \otimes h \in \mathbf{F}_1$ on $\mathscr{B}$. Then $0 = \int \phi(\chi_F - g\bar{h})$ for all $\phi \in \mathscr{B}$. Thus $g\bar{h} = 0$ on $\mathbb{T} \setminus (E \cup F)$. Since $g, h \in H^2$, it follows that $g\bar{h} \equiv 0$. This shows $\hat{\chi}_F \notin \mathscr{B}_\perp + \mathbf{F}_1$ which means $\mathscr{B}$ is not elementary.

Suppose finally that $\mathscr{B} \supset \mathscr{A}$ is a proper ultraweakly closed subspace of $\mathscr{T}$. Because of proper containment, there is a non-zero $f \in L^1$ which is orthogonal to ϕ for each $T_{\phi \in} \mathscr{B}$. In particular, $f \perp H^\infty$, i.e. $f \in H^1_0$. But then Proposition 3.2 tells us that $f = g\bar{h} \in H^2 \overline{H^2}$ whence $g \otimes h$ is a rank one member of $\mathscr{B}_\perp$.

4. HYPERPLANES

We preserve our earlier notation: $\mathscr{T}$ for the space of all Toeplitz operators and $\mathscr{A}$ for the algebra of analytic Toeplitz operators.

Corollary 2.5 tells us that every hyperplane in $\mathscr{T}$ takes the form $\{T_\phi \colon \int \phi f = 0\}$ for some $f \in L^1$. Since $\langle T_\phi, g \otimes h \rangle = \int \phi g\bar{h}$, we see that the general intransitive hyperplane in $\mathscr{T}$ takes the following form.

Definition 4.1. Let $g, h \in H^2 \setminus \{0\}$. Then $\mathscr{B}_{g \otimes h} \equiv \{T_\phi \colon \int \phi g\bar{h} = 0\}$

It is easy to identify the rank one members in the preannihilators of such hyperplanes.

LEMMA 4.2. *In order for the rank one operator* $x \otimes y$ *to belong to the preannihilator of* $\mathscr{B}_{g\otimes h} gh$ *it is necessary and sufficient that* $x\bar{y}$ *be a scalar multiple of* $g\bar{h}$.

Proof. We have $x \otimes y - \lambda g \otimes h \perp \mathscr{F}$ if and only if $\int \phi(x\bar{y} - \lambda g\bar{h}) = 0$ for each $\phi \in L^\infty$. This completes the proof since $(\mathscr{B}_{g\otimes h} gh)_\perp$ is the linear span of $g \otimes h$ and $\mathscr{F}_\perp$.

The following Lemma is the crucial step in establishing Proposition 1.4. To motivate the proof matricially, consider the special case $g = h = 1$. Relative to the standard basis for H^2, the first two rows of the matrix of the operator $z - a \otimes z/(1 - az)$ (for $-1 < a < 1$) are given by

$$\begin{bmatrix} 0 & -a & -a^2 & -a^3 & \cdots \\ 0 & 1 & a & a^2 & \cdots \end{bmatrix},$$

all other rows being zero. Since these rank one operators all belong to the span of $1 \otimes 1$ and $\mathscr{F}_\perp$, successive differentiation and evaluation at $a = 0$ shows that each matrix

$$\begin{bmatrix} \cdots & 0 & -1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & \cdots \end{bmatrix}$$

corresponds to an operator in the closed linear span of $(\mathscr{B}_{1\otimes 1})_\perp \cap \mathbf{F}_1$. But this is just what it takes to show that the first two rows of any operator $B \in \operatorname{ref} \mathscr{B}_{1\otimes 1}$ take the form required of a Toeplitz operator.

LEMMA 4.3. *Suppose* g, h *are outer functions and* B *is an operator on* H^2 *satisfying* $(Bx, y) = 0$ *whenever* $x\bar{y} = g\bar{h}$. *Then* B *is a Toeplitz operator.*

Proof. Write M for the set of trace class operators t satisfying $\langle B, t\rangle = 0$. By hypothesis, every rank one operator $x \otimes y$ for which $x\bar{y} = g\bar{h}$ belongs to M. In particular, for each fixed inner function ϕ, we have

$$(\phi - a)\, \phi^k g \otimes \frac{\phi^{k+1} h}{1 - \bar{a}\phi} \in M, \qquad a \in \mathbb{D}, \quad k = 0, 1, \ldots. \tag{4.1}$$

Expanding this expression as a power series in a yields

$$\phi^{k+1} g \otimes \phi^{k+1} h + \sum_{n=1}^{\infty} a^n (\phi^{k+1} g \otimes \phi^{k+n+1} h - \phi^k g \otimes \phi^{k+n} h) \in M,$$
$$a \in \mathbb{D}, \quad k = 0, 1, \ldots.$$

Successively differentiating and substituting $a=0$ shows that the coefficients of this power series all belong to M. Since we also have $g\otimes h\in M$, this yields

$$\phi^{k+1}g\otimes\phi^{k+n+1}h-\phi^{k}g\otimes\phi^{k+n}h\in M,\qquad n,k=0,1,\dots. \tag{4.2}$$

Direct multiplication also shows that

$$\frac{\phi^{k+1}g}{1-\bar{a}\phi}\otimes(\phi-a)\,\phi^{k}h\in M,\qquad a\in\mathbb{D},\quad k=0,1,\dots.$$

Repeating the argument of the preceding paragraph thus shows that Display 4.2 remains true if we change the places of k and $k+n$. Hence

$$\phi^{k+1}g\otimes\phi^{\ell+1}h-\phi^{k}g\otimes\phi^{\ell}h\in M,\qquad \ell,k=0,1,\dots.$$

Taking linear combinations and specializing to $\phi(z)=z$, we conclude that

$$zpg\otimes zqh-pg\otimes qh\in M,\qquad p,q\ \text{polynomials.}$$

Because g, h are outer functions, we see that

$$Sx\otimes Sy-x\otimes y\in M,\qquad x,y\in H^2,$$

i.e., that

$$(BSx,Sy)=(Bx,y),\qquad x,y\in H^2.$$

Thus $S^*BS=B$, and an appeal to Proposition 2.2 completes the proof.

COROLLARY 4.4. *$\mathcal{B}_{g\otimes h}gh$ is reflexive whenever g and h are outer functions.*

Proof. Let $B\in\operatorname{ref}\mathcal{B}_{g\otimes h}gh$. Lemma 4.2 sets up the hypothesis of Lemma 4.3. Thus $B\in\mathcal{F}$ and $B\perp g\otimes h$, so $B\in\mathcal{B}_{g\otimes h}gh$ by definition.

Lemma 4.6 provides a bridge between Corollary 4.4 and the full force of Proposition 1.4. We first review a well-known general fact.

LEMMA 4.5. *Suppose W and V are bounded operators on* **H** *and* **S** *is a linear manifold of* **B**(**H**). *Then* $W(\operatorname{ref}\mathbf{S})V\subset\operatorname{ref}(W\mathbf{S}V)$.

Proof. Since reflexive closures respect adjoints, it suffices to establish the Lemma in the case $V=I$. Suppose $B\in\operatorname{ref}\mathbf{S}$. Then for each $f\in\mathbf{H}$, we have $Bf\in\overline{\mathbf{S}f}$ which implies $WBf\in\overline{W\mathbf{S}f}$. This means $WB\in\operatorname{ref}(W\mathbf{S})$, as desired.

LEMMA 4.6. *Suppose* $p, q \in H^2$ *and* $A \in \operatorname{ref} \mathscr{B}_{p \otimes q}$. *Then there is a unique Toeplitz operator* T_ψ *having the property that* $T_{\bar{v}} A T_u = T_{\bar{v}} T_\psi T_u$ *whenever* u, v *are inner functions dividing* p, q *respectively and* $\mathscr{B}_{(p/u) \otimes (q/v)}$ *is reflexive. Moreover,* $T_\psi \in \mathscr{B}_{p \otimes q}$

Proof. Note to begin with, that there is at least one pair u, v such that $\mathscr{B}_{(p/u) \otimes (q/v)}$ is reflexive—indeed, Corollary 4.4 tells us that canonical inner factors of p, q, will do.

For uniqueness, suppose $T_{\bar{v}_1} A T_{u_1} = T_{\bar{v}_1} T_{\psi_1} T_{u_1}$ and $T_{\bar{v}_2} A T_{u_2} = T_{\bar{v}_2} T_{\psi_2} T_{u_2}$. Then $T_{\bar{v}_1 \bar{v}_2 \psi_1 u_1 u_2} = T_{\bar{v}_1 \bar{v}_2 \psi_2 u_1 u_2}$. Since the symbol map is injective, this implies $\bar{v}_1 \bar{v}_2 \psi_1 u_1 u_2 = \bar{v}_1 \bar{v}_2 \psi_2 u_1 u_2$, whence $\psi_1 = \psi_2$.

For existence, suppose u, v are any inner functions dividing p, q respectively such that $\mathscr{B}_{(p/u) \otimes (q/v)}$ is reflexive. Since $T_{\bar{v}} T_{\phi T u} = T_{\bar{v} \phi u}$ for all $\phi \in L^\infty$ by [Do, Proposition 7.5], we have $T_{\bar{v}} \mathscr{B}_{p \otimes q} T_u \subset \mathscr{B}_{(p/u) \otimes (q/v)}$. Suppose now that $A \in \operatorname{ref} \mathscr{B}_{p \otimes q}$. Then Lemma 4.5 implies

$$T_{\bar{v}} A T_u \in T_{\bar{v}} (\operatorname{ref} \mathscr{B}_{p \otimes q}) T_u \subset \operatorname{ref} (T_{\bar{v}} \mathscr{B}_{p \otimes q} T_u) \subset \operatorname{ref} \mathscr{B}_{(p/u) \otimes (q/v)}.$$

Thus in fact, $T_{\bar{v}} A T_u = T_\phi \in \mathscr{B}_{(p/u) \otimes (q/v)}$. Take $\psi = v \phi \bar{u}$. Then $T_{\bar{v}} A T_u = T_{\bar{v} \psi u} = T_{\bar{v}} T_\psi T_u$ as desired. Since $T_\phi = T_{\bar{v} \psi u} \in \mathscr{B}_{(p/u) \otimes (q/v)}$, a direct calculation shows that $T_\psi \in \mathscr{B}_{p \otimes q}$.

All the pieces are now in place for our proof of Proposition 1.4.

PROPOSITION 1.4. *Every ultraweakly closed, intransitive hyperplane in* $\mathscr{F}$ *is reflexive.*

Proof. We must show $\mathscr{B}_{ug \otimes vh}$ is reflexive whenever u, v are inner and g, h are outer.

Suppose first that $v = 1$. Given $A \in \operatorname{ref} \mathscr{B}_{ug \otimes h}$, apply Lemma 4.6 (with $p = ug$ and $q = h$) to find $T_\psi \in \mathscr{B}_{ug \otimes h}$ with $A T_u = T_\psi T_u$, and take $B = A - T_\psi$. Suppose $x, y \in H^2$ with $x \bar{y} = g \bar{h}$. Then direct multiplication shows that

$$(u - b) x \otimes \frac{y}{1 - \bar{b} u} \perp \mathscr{B}_{ug \otimes h}, \qquad b \in \mathbb{D}.$$

Because $B \in \operatorname{ref} \mathscr{B}_{ug \otimes h}$, it must be orthogonal to all of these rank one operators; since $B T_u = 0$, we have

$$-b \left(Bx, \frac{y}{1 - \bar{b} u} \right) = 0, \qquad b \in \mathbb{D}.$$

Dividing by b, and then setting $b = 0$, we conclude that

$$(Bx, y) = 0, \qquad \text{for all} \quad x, y \in H^2 \quad \text{with} \quad x\bar{y} = g\bar{h}.$$

In view of Lemma 4.3, we conclude that B and A are Toeplitz operators. Since $A \perp ug \otimes h$, it follows that $A \in \mathscr{B}_{ug \otimes h}$ and we have shown that $\mathscr{B}_{ug \otimes h}$ is reflexive.

Since taking adjoints preserves reflexivity, the argument of the preceding paragraph also shows that $\mathscr{B}_{g \otimes vh}$ is reflexive for each inner function v.

Finally, let u, v be arbitrary inner functions, and suppose $A \in \operatorname{ref} \mathscr{B}_{ug \otimes vh}$. Since we now know that both $\mathscr{B}_{ug \otimes h}$ and $\mathscr{B}_{g \otimes vh}$ are reflexive, Lemma 4.6 yields a Toeplitz operator $T_\psi \in \mathscr{B}_{ug \otimes vh}$ *simultaneously* satisfying $AT_u = T_\psi T_u$ and $T_{\bar{v}} A = T_{\bar{v}} T_\psi$. Take $B = A - T_\psi$. Suppose x, $y \in H^2$ with $x\bar{y} = g\bar{h}$. Then direct multiplication shows that

$$\frac{(u-b)x}{1-\bar{c}v} \otimes \frac{(v-c)y}{1-\bar{b}u} \perp \mathscr{B}_{ug \otimes vh}, \qquad b \in \mathbb{D}.$$

Because $B \in \operatorname{ref} \mathscr{B}_{ug \otimes vh}$, it must be orthogonal to all of these rank one operators; since $BT_u = T_{\bar{v}} B = 0$, we have

$$b\bar{c}\left(B\left(\frac{x}{1-\bar{c}v}\right), \frac{y}{1-\bar{b}u}\right) = 0, \qquad b, c \in \mathbb{D}.$$

Dividing by $b\bar{c}$, and then setting $b = c = 0$, we conclude that

$$(Bx, y) = 0, \qquad \text{for all} \quad x, y \in H^2 \quad \text{with} \quad x\bar{y} = g\bar{h}.$$

In view of Lemma 4.3, we conclude that B and A are Toeplitz operators, whence $A \in \mathscr{B}_{ug \otimes vh}$ and we have established reflexivity of $\mathscr{B}_{ug \otimes vh}$.

We are now ready to prove the main results of the paper.

THEOREM 1.1″. *Suppose $\mathscr{B}$ is a linear space of Toeplitz operators which is closed in the ultraweak operator topology. Then the following are equivalent.*

(1) *$\mathscr{B}$ is intransitive.*

(2) *There are functions $g, h \in H^2 \backslash \{0\}$ such that $(Bg, h) = 0$ for each $B \in \mathscr{B}$.*

(3) *There is an integrable function f on the unit circle such that* $\log |f|$ *is also integrable and $\int f\phi = 0$ whenever ϕ is the symbol of a member of $\mathscr{B}$.*

(4) *$\mathscr{B}$ is reflexive.*

THEOREM 1.2. *Every proper, ultraweakly closed subspace of $\mathscr{T}$ which contains $\mathscr{A}$ is reflexive.*

Proof. For 1.1″, note first that (1) iff (2) by definition, (2) iff (3) by Bourgain's characterization 3.2, and $(4) \Rightarrow (1)$ is trivial. Finally, if (2) holds, then $[g \otimes h]^{\perp} \cap \mathscr{F}$ is an intransitive hyperplane in $\mathscr{F}$. Propositions 1.3 and 1.4 tell us this hyperspace is hereditarily reflexive, whence its subspace $\mathscr{B}$ must also be reflexive.

For 1.2, suppose $\mathscr{A} \subset \mathscr{B} \subsetneq \mathscr{F}$. Apply Corollary 3.5(4) to conclude that $\mathscr{B}$ is intransitive, and appeal to the preceding result.

5. EXAMPLES, QUESTIONS, AND COMMENTS

Example 5.1. No finite-dimensional space of Toeplitz operators can be transitive, so they must all be reflexive by Theorem 1.1. Similarly, Theorem 1.2 shows that every finite-dimensional extension of $\mathscr{A}$ in $\mathscr{F}$ must also be reflexive.

Example 5.2. It is an immediate application of Theorem 1.2 that the set of Toeplitz operators whose matrices have zeros on a fixed superdiagonal is reflexive.

Example 5.3. Let $f \in L^{\infty} \backslash \{0\}$ and consider the space $T_f \mathscr{A}$. This is $\xi(fH^{\infty})$, the space of Toeplitz operators whose symbols belong to fH^{∞}.

(1) If $\bar{f}$ is inner, then $T_f \mathscr{A} \supset T_f T_{\bar{f}} \mathscr{A} = \mathscr{A}$ so Theorem 1.2 still applies to show that $T_f \mathscr{A}$ is reflexive.

(2) More generally, suppose $1/f \in L^{\infty}$. Note first that fH^{∞} is weak-star closed, so $T_f \mathscr{A}$ is at least ultraweakly closed. Moreover, since $\log |f|$ is integrable, we can write $z/f = g\bar{h}$ for some $g, h \in H^2$. For any $\phi \in H^{\infty}$, we have

$$< T_f T_{\phi}, g \otimes h > = \int f \phi g \bar{h} \, dm = \int z \phi(z) \, dm(z) = 0,$$

so $T_f \mathscr{A}$ is intransitive and Theorem 1.1 tells us it must be reflexive.

(3) If it is only known that $1/f \in L^1$, it is still true that $\log |f|$ is integrable and hence that $T_f \mathscr{A}$ is intransitive, and its ultraweak closure is reflexive. However, $T_f \mathscr{A}$ can fail to be ultraweakly closed even when $f \in H^{\infty}$. Indeed, if $f(z) = 1 - z$, then $\{1 - z^n\}_{n=1}^{\infty}$ is a sequence in fH^{∞} whose weak-star limit, the constant function 1, fails to belong to that space.

(4) If $\log |f|$ is not integrable, then $T_f \mathscr{A}$ is transitive. Indeed, if $g \otimes h$ belonged to the preannihilator of this space, we would have $\int \phi f g \bar{h} = 0$ for every $\phi \in H^{\infty}$. But this would force $fg\bar{h} \in H_0^1$ which is impossible since $\log |fg\bar{h}| = \log |f| + \log |g| + \log |h|$ cannot be integrable.

In particular, $T_f H^\infty$ is transitive whenever f is a non-trivial characteristic function.

EXAMPLE 5.4. *Let* $\mathbf{\Psi}$ *be a family of Blaschke products and write* M *for the linear span of* $\bar{\Psi} H^\infty$. *Also, for each* $a \in \mathbb{D}$ *write* n_a *for the maximal power of* $z - a$ *dividing some member of* $\mathbf{\Psi}$.

(1) *If* $\sum n_a(1 - |a|) < \infty$ *then the weak-star closure of* $\xi(M)$ *is reflexive.*

(2) *If* $\sum n_a(1 - |a|) = \infty$ *then* M *is weak-star dense in* L^∞ *and* $\xi(M)$ *is transitive.*

Proof. Write Λ for the set of zeros of the members of Ψ. For (1), note that the hypothesis implies the existence of a Blaschke product ψ_0 having a zero of order n_a at each $a \in \Lambda$. Part (2) of the preceding example tells us that $\xi(\bar{\psi}_0 H^\infty)$ is reflexive and elementary, so the proof is completed by observing that $\xi(M)$ is contained in this space.

To establish (2), note first that $M \supset H^\infty$, whence $H^\infty_\perp \supset M_\perp$. Take $f \in M_\perp \subset L^1$. Thus $f \in H^\infty_\perp$ and f is a holomorphic function vanishing at 0 ([BFP] p. 33). Hence $f = z f_1$ for some function $f_1 \in H^1$. Fix $a \in \Lambda$ and $j \leqslant n_a$ for the moment. Writing B_a for the Blaschke factor $(z - a)/(1 - \bar{a}z)$, note that the hypothesis implies that B_a^j divides some member of Ψ, whence its conjugate belongs to M. Since $f = z f_1 \in M_\perp$, we have

$$0 = \langle \bar{B}_a^j, f \rangle = \int \frac{z f_1(z)(1 - \bar{a}z)^j}{(z - a)^j}\, dm = \frac{1}{2\pi i} \int_{|z|=1} \frac{f_1(z)(1 - \bar{a}z)^j}{(z - a)^j}\, dz. \tag{5.1}$$

Suppose we know that some power $(z - a)^k$ divides $f_1(z)$. Then we can factor $f_1(z) = (z - a)^k g(z)$. If $k < n_a$, then we can apply Eq. 5.1 with $j = k + 1$ to get

$$\int_{|z|=1} \frac{g(z)(1 - \bar{a}z)^{k+1}}{z - a}\, dz = 0.$$

But then Cauchy's Theorem would imply that $z - a$ divides $g(z)$ whence $(z - a)^{k+1}$ actually divides f_1. This inductive argument shows that f_1 has a zero of order (at least) n_a at each $a \in \Lambda$. In view of the Corollary on Page 18 of [Du], (zeros being repeated according to multiplicity in that result), we conclude that $f_1 \equiv 0$. Thus we have shown that $M_\perp = \{0\}$ and we have weak-star density of M in L^∞. This in turn yields weak-star density of $\xi(M)$ in $\mathscr{F}$ and the proof is completed by appealing to Theorem 3.1.

The transitive spaces exhibited in the last example are in fact weak-star-dense in $\mathscr{F}$. See Corollary 3.5 above for others.

EXAMPLE 5.5. This is a good place to mention the finite-dimensional analogue of $\mathscr{F}$. As detailed in [Az, Example 3.5], like $\mathscr{F}$, the full space $\mathscr{F}_n$ of $n \times n$ Toeplitz matrices is transitive and 2-reflexive. In one sense, it is better behaved than $\mathscr{F}$, being elementary and having no proper transitive subspaces. (Compare with Corollary 3.5.) On the other hand, the algebra $\mathscr{A}_n$ generated by the shift on $\mathbb{C}^n$ is neither transitive nor reflexive, so the analogues of Sarason's result, Theorems 1.1 and 1.2, and Proposition 1.4 all fail in this setting.

We turn now to some open questions. We begin by asking whether the converse of Proposition 1.3 is valid.

QUESTION 5.6. *If an ultraweakly closed subspace of* $\mathscr{F}$ *is elementary, must it be reflexive.*

QUESTION 5.7. *Are the reflexive spaces exhibited in this paper hyperreflexive*?

W. Arveson's notion of hyperreflexivity is a metric version of reflexivity. K. Davidson [Da] has shown that $\mathscr{A}$ is hyperreflexive with constant 19, but we have not examined whether this carries over to the spaces of Theorem 4.3 nor whether $\mathscr{F}$ is 2-hyperreflexive.

There is also a metric version of Definition 1.6.

DEFINITION 5.8. Given $k \in \mathbb{N}$ and $r \geqslant 0$, we say a linear manifold $\mathbf{S}$ in $\mathbf{B}(\mathbf{H})$ has *Property* $\mathbb{A}_{1/k}(r)$ if every $t \in \mathbf{T}$ having $\|t\|_1 < 1$ can be perturbed by a member of $\mathbf{S}_\perp$ to yield a member of $\mathbf{F}_k$ of trace norm at most r.

QUESTION 5.9. *What are best possible metric versions of the results concerning elementary spaces proved in the body of this paper*?

It is clear that any space enjoying Property $\mathbb{A}_{1/k}(r)$ for some r must be k-elementary. While the converse fails, k-elementary does imply Property $\mathbb{A}_{1/2k}(r)$ for some r. (See [Az, Section 10] for an expository account of this.) Thus for example, Corollary 3.5 implies that $\mathscr{F}$ has Property $\mathbb{A}_{1/4}(r)$ for some r. Actually, we can prove that $\mathscr{F}$ has Property $\mathbb{A}_{1/3}(6)$, but we don't know whether it has Property $\mathbb{A}_{1/2}(r)$ for some r. In a similar vein, we can show that the spaces of Example 5.3(2) have Property $\mathbb{A}_1(1)$, but we don't know whether all reflexive subspaces of $\mathscr{F}$ have some Property $\mathbb{A}_1(r)$; this would require a metric examination of [Bo].

The Jordan operators studied in [Be] should also be mentioned in this connection. The *Jordan block* $S(\theta)$ associated with the inner function θ acts on the Hilbert space $\mathscr{H}(\theta) = H^2 \ominus \theta H^2$ by $S(\theta) = P_{\mathscr{H}(\theta)} S|_{\mathscr{H}(\theta)}$; the classical finite dimensional shift of Example 5.5 corresponds to $\theta(z) = z^{n-1}$. The Jordan operators of [Be] are (special) direct sums of such blocks.

Proposition 3.1.21 of [Be] shows these operators all have Property $\mathbb{A}_1(1)$ and recent work of V. V. Kapustin [K] characterizes the reflexive operators in this class.

We conclude with an observation concerning the relation between weak-star and ultraweak topologies.

Remark 5.10. Parts (1) and (2) of Corollary 2.3 apply to any isometry from L^∞ into $\mathbf{B}(\mathbf{H})$. To see that (3) and (4) can fail at this level of generality, let $\{E_n\}$ be an infinite partition of $\mathbb{T}$ into sets of strictly positive measure. The collection

$$\left\{\phi \in L^\infty : \lim_{n\to\infty} \int_{E_n} \phi \, dm = 0\right\}$$

is a norm closed ideal in L^∞ which is dense in the weak-star topology. (It is analogous to c_0 in ℓ_∞.) Choose a multiplicative linear functional θ on L^∞ whose kernel contains this ideal. Thus there is a net $\{\phi_\alpha\}$ which converges weak-star to 0 with $\theta(\phi_\alpha) \equiv 1$.

Now let $\rho: L^\infty \to \mathbf{B}(L^2 \oplus \mathbb{C})$ by taking $\rho(\phi)$ to be multiplication by $\phi \oplus \theta(\phi)$. Then ρ is (multiplicative and) isometric, but while the net $\rho(\phi_\alpha)$ converges to $0 \oplus 0$ weak-star, its ultraweak limit is $0 \oplus 1$.

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