WANDERING SETS FOR A CLASS OF BOREL ISOMORPHISMS OF $[0,1)$

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Abstract. A wandering set for a map ϕ is a set containing precisely one element from each orbit of ϕ. We study the existence of Borel wandering sets for piecewise linear isomorphisms. Such sets need not exist even when the parameters involved are rational, but they do exist if in addition all the slopes are powers of 2. For ϕ having at most one discontinuity, the existence of a Borel wandering set is equivalent to rationality of the Poincaré rotation number. We compute the rotation numbers for a special class of such functions. The main result provides a concrete method of connecting certain pairs of wavelet sets.

1. Introduction

Let h be a bijective map of a set X onto itself. A subset S of X is *invariant under* h if $h(S) = S$. The *orbit* $[x]$ of a point $x \in X$ is the smallest invariant subset of X containing x. We say $x \in X$ is *periodic* if its orbit is finite. A *wandering set for* h is a subset W of X containing precisely one point from each orbit. Our usage differs from the standard terminology of [18] (and [2]) in that we do not require the sets $\{h^{(n)}(W)\}_{n\in\mathbf{Z}}$ to be disjoint, but we do require them to exhaust X; the present convention allows for a unified treatment of periodic and non-periodic points in the statements of our main results.

In this paper, we study the existence of Borel wandering sets for a class of Borel measurable maps. Our interest in this problem arose in connection with the connectivity question for the set of orthonormal wavelets. Related problems come up in a variety of contexts ranging from dynamical systems theory to descriptive set theory.

We focus attention on piecewise linear isomorphisms of $[0,1)$; these are called *affine interval exchange maps* in [3] and [2]. The classical example is the "rotation" map R_t which sends $x \in [0,1)$ to the fractional part of $x+t$. If $t = \frac{m}{n}$, with m, n relatively prime integers, then $[0, \frac{1}{n})$ provides a wandering set for R_t. On the other hand, when t is irrational, then every orbit under R_t is infinite, and translation invariance of Lebesgue measure shows that R_t does not even have a Lebesgue measurable wandering set.

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In [1], M. Boshernitzan examined modifications of the classical maps R_t obtained by varying the slopes of their components. Somewhat surprisingly, rationality of the parameters involved is neither necessary nor sufficient for the existence of a Borel wandering set.

Proposition 1.1. *Suppose h is a discontinuous bijection of $[0,1)$ whose graph consists of two line segments having positive slopes $p, q \neq 1$. Then in order for h to have a Borel wandering set it is necessary and sufficient that $\frac{\ln p}{\ln q}$ be rational.*

In particular, Borel wandering sets exist if p and q are powers of 2. Most of the present paper is devoted to generalizing this positive result to the multi–slope setting.

Theorem 1.2. *Suppose h is a bijection of $[0,1)$ whose graph consists of finitely many line segments, with all discontinuities and changes of slope occurring at rationals. If all slopes involved are powers of 2, then h has a wandering set which is a finite union of intervals W_i (some of which may be singleton sets).*

It would be interesting to know the full fate of Proposition 1.1 in the multislope case.

A classical result of J. Glimm and E. Effros states that a homeomorphism acting on a complete separable metric space X will admit a Borel wandering set if and only if no $x \in X$ is a cluster point of its orbit. One strategy for applying this to Theorem 1.2 would be to throw away the orbits of all points at which h is discontinuous; the remaining set, being of class G_δ in $[0,1]$, is metrizable by a complete separable metric. Another technique would be to double the offending points as in [13].

Each of these approaches would necessitate examining uncountably many orbits under h. The hypothesis of piecewise linearity, on the other hand, suggests that everything should be determined by looking at the orbits of the finite set A of points at which h either changes slope or fails to be continuous. This is the tack we take in proving Theorem 1.2. The assumption that the slopes are all powers of 2 gives us control on the union A_∞ of orbits of points in A. The key Lemma 4.1 states that if $a \in \mathbb{Q}$ is not periodic, then there exists a closest member of $A_\infty \cup \{1\}$ to the right of it. The resulting proof of Theorem 1.2 is effective in the sense that the intervals it describes are built from A, the (necessarily finite) set of limit points of A_∞, and points generated by Lemma 4.1. It also provides global dynamical information.

Corollary 1.3. *For h and W_i as in Theorem 1.2, it can be arranged that within a given interval W_i, either all points have the same period, or their orbits share a common cluster point set.*

The structure of the remainder of the paper is as follows. In Section 2, we review Poincaré rotation numbers and their relation to the Glimm–Effros Theorem. The examples of Section 3 motivate the statement of Theorem 1.2.

Section 4 proves the existence of a Borel wandering set when the function h of Theorem 1.2 has a single discontinuity; appeal to Theorems of Poincaré and Denjoy make this argument relatively short. By contrast, the longer proof for the general case given in Section 5 is essentially self–contained. The strategy of focusing on break points works in a more general setting.

Proposition 1.4. *Suppose h is a bijection of an interval whose discontinuity set is finite. If the orbit of each discontinuity of h has only countably many cluster points, then h admits a Borel wandering set.*

Section 6 applies Theorem 1.2 to establish the existence of continuous paths of wavelets (via Theorem 2 in [14]) connecting (MFS)-wavelets whose Fourier transforms are supported on sets which are finite unions of intervals whose endpoints are rational multiples of π. We take the opportunity to illustrate the constructive nature of Theorem 1.2 and Corollary 1.3 on a particular wavelet–induced map.

The abstract argument of Section 4 does not compute the rotation numbers of the maps in question. In the final section of the paper, we use renormalization techniques to develop a recursive procedure for computing the rotation numbers of all composite maps of the form $R_t \circ \varphi$, where φ is the continuous bijection on $[0,1)$ whose graph consists of line segments having slopes 2 and $\frac{1}{2}$. This discussion is independent of Sections 4 through 6.

2. Preliminaries

Let $(X,\mathcal{B})$ be a Borel space and $h : X \to X$ be a Borel isomorphism, i.e., h is a bijection and h and h^{-1} are Borel measurable maps. We denote by $h^{(k)}$ the identity function on X when $k = 0$ and the composition of h [resp. h^{-1}] with itself $|k|$ times when $k > 0$ [resp. $k < 0$]. The *orbit* of a point $x \in X$ under the action of h is the set $[x] := \{h^{(k)}(x);\ k \in \mathbb{Z}\}$. More generally, the *saturation* A_∞ of a subset A of X is defined by $A_\infty := \bigcup_{j\in\mathbb{Z}} h^{(j)}(A)$; it is the union of the orbits of the members of A.

Definition 2.1. A set $\Omega \subset X$ is called *invariant* under h if $h(\Omega) = \Omega$. A set $W \subset X$ is said to be a *wandering set* for h if W contains one and only one point of each orbit $[x]$, $x \in X$.

As we have already mentioned, we are interested in wandering sets which are Borel sets with respect to the standard Borel structure on $[0,1)$. The notion of wandering set has been considered and studied previously in various papers and from different points of view. For a self-contained well-written monograph on this subject and related topics we recommend Nadkarni's book [18]. It was noted in the Introduction that Definition 2.1 differs from Nadkarni's use of the term. However, it is proved in [18] that there is always a Borel set W_0 containing precisely one point from each periodic orbit of h, whence existence of a Borel set satisfying Definition 2.1 is equivalent to the existence of a Borel set W_1 such that $\{h^{(n)}(W_1)\}_{n\in\mathbb{Z}}$ form a partition (without repetition) of $\{x \in [0,1) : x \text{ is not periodic for } h\}$.

From the dynamical systems point of view, isomorphisms for which wandering sets exist are considered very simple (for instance, they are called *elementary Borel automorphisms* in [18] or *smooth* in [5]). In fact, this property is equivalent to some "non-ergodicity" conditions as they are called by E. Effros in [6], improving upon an earlier paper of Glimm [8]. Using these equivalent conditions essentially proved in [6] and improved in [7], one obtains very general criteria for the existence of Borel wandering sets for homeomorphisms on complete separable metric spaces (Theorem 2.3 below). One important result in this theory which makes Theorem 2.3 applicable to Borel isomorphisms on standard Borel spaces is the following result due to Ramsay and Mackey (see [21] and [22]).

Theorem 2.2. *If h is a Borel isomorphism acting on a standard Borel space $(X, \mathcal{B})$ then there is a complete separable metric topology which generates $\mathcal{B}$ and with respect to which h becomes a homeomorphism.*

If h is a homeomorphism acting on a complete separable metric space X and $x \in X$ we say that x is *recurrent* if $h^{(k)}(x) \neq x$ for all $k \in \mathbb{Z}$ and $h^{(n_k)}(x) \to x$ for some sequence $\{n_k\}$ in $\mathbb{Z}$ ($|n_k| \to \infty$); this is equivalent to requiring that x be a cluster point of its orbit. A Borel measure ν on X is *ergodic* with respect to h if for every invariant set Ω under h either $\nu(\Omega) = 0$ or $\nu(X\backslash\Omega) = 0$. A point $x_0 \in X$ is a *periodic point* of period $n \in \mathbb{N}$ for h if $h^{(n)}(x_0) = x_0$ and $h^{(k)}(x_0) \neq x_0$ for all $k < n$ ($k \geq 1$). The next theorem is one of best known results in the theory of Borel isomorphisms.

Theorem 2.3 (Glimm-Effros). *Let h be a homeomorphism acting on a complete separable metric space X. The following are equivalent:*

(i) *There exists a Borel wandering set for h.*
(ii) *There is no recurrent point for h.*
(iii) *There is no non–atomic measure on the Borel subsets of X which is ergodic with respect to h.*

For more general results along these lines we refer the reader to [5], [10], [12], and [15].

Throughout this paper, "piecewise" means in *finitely* many pieces; thus piecewise continuous functions are only allowed finitely many discontinuities and piecewise linear functions may only involve finitely many slopes. We are interested in studying the existence of Borel or measurable wandering sets for maps h which are piecewise linear isomorphisms of the unit interval $I := [0,1)$ which are right-continuous at every point and such that the various slopes are positive real numbers; these are referred to as affine interval exchange maps in the literature. We denote this class by $\mathcal{C}_0$. In spite of the fact that there is substantial work on classifying Borel isomorphisms in general and the above characterization, when confronted with concrete classes of maps the existence of Borel wandering sets is not always easy to establish.

One particular subclass, $\mathcal{C}_1$, of $\mathcal{C}_0$ which will be considered in our paper consists of those isomorphisms are piecewise increasing and have only one point of discontinuity. Every map $\widetilde{h} \in \mathcal{C}_1$ can be viewed as a homeomorphism of the unit circle S^1. Indeed, by conjugating with $\varphi(t) := \exp(2\pi i t),\ t \in [0,1)$, we remove the discontinuity, obtaining a homeomorphism $h = \varphi \circ \widetilde{h} \circ \varphi^{-1}$ of the unit circle S^1. Conversely, if h is an orientation preserving homeomorphism of S^1 then we can associate an isomorphism $\widetilde{h} = \varphi^{-1} \circ h \circ \varphi$ of I which is continuous except possibly at one point denoted here for convenience by c_h ($c_h = 0$ if $h(1) = 1$). It is convenient to carry over terminology between these representations, e.g., to call h piecewise continuous if $\widetilde{h}$ is.

The following facts from the dynamical system theory of homeomorphisms of the unit circle will be needed later. These results can be found in [11], [19], and [20]. Let us consider an orientation preserving homeomorphism h of S^1, and let $\widetilde{h}$ and $c := c_h$ be as above. One can define the *rotation number*, r_h, for such an h as being the unique real number r_h in $(0,1]$ such that

$$r_h = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{[c,1)}(\widetilde{h}^{(k)}(x)), \quad (\mathrm{mod}\ 1) \tag{1}$$

for some $x \in I$. It turns out that r_h is well defined (the right hand side of (1) is independent of the choice of $x \in I$), it is invariant under conjugation and continuous (mod 1) under perturbations [19]. The following result is a remarkable fact in this theory.

Theorem 2.4 (**Poincaré**). *Let h be an orientation preserving homeomorphism of the unit circle and $\widetilde{h}$ be the map associated to h as above. Then h has a periodic point if and only if r_h is a rational number and in this case every orbit is asymptotically periodic* (*i.e. the cluster points of every orbit form a periodic orbit*)*. If the rotation number r_h is irrational, then there exits a unique continuous surjective monotone increasing map $\psi : I \to I$ such that*

$$\psi \circ \widetilde{h} = R \circ \psi, \tag{2}$$

where $R(x) = x + r_h$ (mod 1), $x \in I$. In the later case, the orbit of every point is either dense in I (*when ψ is a homeomorphism*) *or the set of its cluster points is a Cantor type set in I which is independent of the orbit chosen.*

Combining Theorem 2.3 and Theorem 2.4 one obtains the following simple criterion.

Proposition 2.5. *The following are equivalent for an orientation preserving homeomorphism h of the circle:*

(i) *h has a Borel wandering set.*
(ii) *h has a periodic point.*
(iii) *The rotation number of h is rational.*

A well-known theorem of A. Denjoy provides sufficient conditions for the map ψ in (2) to be a homeomorphism. We will make use of a special case of the version of that result presented on Page 76 of [11].

Theorem 2.6 (Denjoy). *Suppose $h : S^1 \to S^1$ is either a piecewise linear homeomorphism or a C^1 diffeomorphism whose derivative has bounded variation. If the rotation number of h is irrational, then h is conjugate to a rotation (i.e. the map ψ in (2) is a homeomorphism).*

3. Examples

The existence of Borel wandering sets for homeomorphisms of intervals provides a simple illustration of the preceding theory. It is instructive to examine the structure of these sets directly.

Proposition 3.1. *Let h be a homeomorphism of $[0,1)$ and write F for its set of fixed points.*

(i) *h has a wandering set of class $\mathfrak{F}_\sigma$.*
(ii) *If h has a wandering interval, then h is the identity map.*
(iii) *In order for h to admit a wandering set which is a finite union of intervals, it is necessary and sufficient that F be a finite union of intervals.*
(iv) *If h is piecewise linear, then it has a wandering set which is a finite union of intervals.*

Proof. For (i), note that if $h(c) > c$ $\{h(c) < c\}$, then the images of $I_c := [c, h(c))$ {respectively $I_c := [h(c), c)$} under the iterates of h form a partition of the component of $[0,1) - F$ containing c. Thus we get a wandering set W for h by choosing one point c from each component of $[0,1) - F$ and taking the union of F with the corresponding intervals I_c. As we consider singleton sets to be intervals, this construction establishes sufficiency of (iii) and (iv) follows as a special case.

There is of course a large degree of arbitrariness in the above construction, but F must be included in W and the boundary of F must be part of the boundary of W. This observation justifies (ii) and necessity of (iii). ∎

We continue with three examples of isomorphisms in $\mathcal{C}_0$ for which there are no measurable wandering sets. These examples will lead us to reasonable conditions to impose on maps in $\mathcal{C}_0$ in order to ensure existence of Borel wandering sets. The first two are classical examples.

For each real number t, define R_t be the map on $[0,1)$ which sends $x \in [0,1)$ to the fractional part of $x+t$. It follows from the definition that the rotation number of the associated circle map is in fact t. Thus Proposition 2.5 tells us that R_t has a Borel wandering set if and only if t is rational. This is also easy to see directly. Indeed, if t is rational with reduced form p/q then the interval $[0,1/q)$ provides the desired wandering set. Suppose conversly that t is irrational, and R_t had a Lebesgue measurable wandering set W. Then the sets $\{R_t^{(k)}(W)\}_{k\in\mathbb{Z}}$ would be pairwise disjoint and $\sum_{k\in\mathbb{Z}} \mu(R_t^{(k)}(W)) = 1$, where μ is Lebesgue measure on $[0,1)$. But, since μ is invariant under translations, $\mu(R_t^{(k)}(W)) = \mu(W)$ for all k which contradicts the above equality.

The second example is the so called the odometer map (cf. [18]) defined by $h(x) = x - 1 + \frac{3}{2^{(n+1)}}$, if $x \in [\frac{2^n-1}{2^n}, \frac{2^{(n+1)}-1}{2^{(n+1)}})$, $n \in \mathbb{N} \cup \{0\}$. As in the preceding example, there are no finite orbits and Lebesgue measure is invariant (and ergodic) under h.

These examples suggest that one needs to impose some conditions on the isomorphism in order to admit even a measurable wandering set. We will focus our study then on isomorphisms h which are piecewise linear maps with finitely many pieces and for which the slopes and breakpoints are rational numbers. The next somewhat surprising result shows that even these conditions are not sufficient to guarantee existence of measurable wandering sets. The following proof is due to M. Boshernitzan [1]; an alternate treatment (closer to the present authors' original argument) can be found in [3].

Proposition 3.2. *Let h be a piecewise linear bijection of $[0,1)$ having two pieces with distinct positive slopes $p, \frac{1}{q}$ respectively, and one point of discontinuity. Then h is conjugate to the rotation R_t where $t = \frac{\ln p}{\ln(pq)}$. Moreover, the measure $d\nu(x) = \frac{1}{1+(pq-1)x} d\mu(x)$ is invariant under h. In particular, h admits a wandering set if and only if the logarithms of p and q are rationally dependent.*

Proof. By conjugating with (the μ-measure preserving isomorphism) $\theta(x) = 1-x$ we can assume that the first piece of h has slope $p > 1$ and so we can explicitly write h as:

$$h(x) = \begin{cases} px + \dfrac{p-1}{pq-1}, & x \in \left[0, \dfrac{q-1}{pq-1}\right), \\ \dfrac{x}{q} - \dfrac{q-1}{q(pq-1)}, & x \in \left[\dfrac{q-1}{pq-1}, 1\right). \end{cases} \tag{3}$$

Now set $\psi(x) = \dfrac{\ln(1+(pq-1)x)}{\ln(pq)}$. A direct computation shows that $\psi h = R_t \psi$, whence h and R_t are indeed conjugate. It follows that the measure $\nu := \psi^{-1} * \mu = \psi' \mu$ is invariant for h. The final assertion of the proposition is a consequence of the fact that conjugate maps have the same rotation number. ∎

In particular, the simple piecewise linear map $h : [0,1) \to [0,1)$ defined by

$$h(s) = \begin{cases} 3s + 2/5, & s \in [0,1/5), \\ s/2 - 1/10, & s \in [1/5,1). \end{cases}$$

does not have a measurable wandering set, even though all parameters involved are rational.

It would be interesting to know the full generalization of Proposition 3.2 to the multi-slope case. In the meantime, we restrict attention to piecewise continuous isomorphisms of $[0,1)$ for which all discontinuities and changes of slope occur at rational points and all slopes involved are integral powers of some fixed positive base which will be taken to be 2 for definiteness. We will denote this class of maps by $\mathcal{C}_2$. The conjugacy trick of the last proof will not work at this level. Indeed, while any map conjugate to a rational rotation must have finite order, most of the maps covered by Theorem 1.2 have non-periodic points.

4. SINGLE DISCONTINUITY CASE

Working in $\mathcal{C}_1 \cap \mathcal{C}_2$ puts Denjoy's Theorem at our disposal. The following key lemma, however, is valid for all $h \in \mathcal{C}_2$. To set the notation, we assume that $h \in \mathcal{C}_2$ is defined by $L (L \in \mathbb{N})$ linear equations $h(x) = 2^{\beta_j}x + \frac{\gamma_j}{M}$, if $x \in I_j := [a_j, a_{j+1})$, $j \in \{1,2,...,L\}$, $M \in \mathbb{N}$, $\beta_j, \gamma_j \in \mathbb{Z}$ and a_j are fractions with denominator M such that the family of intervals $\{I_j\}_{j=1...L}$ is a partition of $[0,1)$. We denote by A the set of the break points $a_1,\ a_2, ...,\ a_L$. Recall that $A_\infty := \bigcup_{n\in\mathbb{Z}} h^{(n)}(A)$ is the union of the orbits of the members of A.

LEMMA 4.1. *Let $h \in \mathcal{C}_2$ and $x_0 \in I$ be a rational number which is not a periodic point for h. Then there is a closest point $c \in A_\infty \cup \{1\}$ to the right of x_0. Moreover, each iterate of h is linear on the interval $[x_0, c)$.*

PROOF. For $k \in \mathbb{Z}$, we set $x_k = h^{(k)}(x_0)$, and write $\varphi(k)$ for the unique number in $\{1,2,...,L\}$ satisfying $x_k \in I_{\varphi(k)}$. Since $x_{k+1} = h(x_k) = 2^{\beta_{\varphi(k)}}x_k + \gamma_{\varphi(k)}/M$ for $k \in \mathbb{N} \cup \{0\}$, it follows inductively that x_k is a rational number which can be written uniquely as $\frac{L_k}{2^{n_k}M'}$ where $L_k \in \mathbb{Z}$, $n_k \in \mathbb{N} \cup \{0\}$, $gcd(L_k, 2) = 1$ and M' is the least common multiple of M and the denominator of x_0. We claim that $\{n_k\}_{k\in\mathbb{N}}$ must converge to infinity. Indeed, let us suppose there is a subsequence $\{n_{k_l}\}_{l\in\mathbb{N}}$ which is bounded by some fixed number $C (C \in \mathbb{N})$. Since there are only finitely many numbers in $[0,1)$ which can be written as $\frac{L}{2^j M'}$, $j \in \{1,2,...,C\}$, $L \in \mathbb{Z}$, there must be some $l, l' \in \mathbb{N}$ $(l \neq l')$ such that $x_{k_l} = x_{k'_l}$ and hence we contradict the fact that $\{x_k\}$ is not periodic.

Since $n_k \to \infty$, there exists a $k_0 \in \mathbb{N}$ such that $n_k > \beta$ for all $k \geq k_0$. For $k \geq k_0$, since $x_{k+1} = 2^{\beta_{\varphi(k)}} x_k + \gamma_{\varphi(k)}/M$ we obtain that

$$x_{k+1} = \frac{L_{k+1}}{2^{n_{k+1}}M'} = \frac{2^{\beta_{\varphi(k)}} L_k}{2^{n_k}M'} + \frac{\gamma_{\varphi(k)}}{M} = \frac{L_k + \gamma_{\varphi(k)} M'' 2^{n_k - \beta_{\varphi(k)}}}{2^{n_k - \beta_{\varphi(k)}} M'},$$

where $M' = MM''$. This implies that $n_{k+1} = n_k - \beta_{\varphi(k)}$ for $k \geq k_0$ and therefore

$$n_k = n_{k_0} - \sum_{j=k_0}^{k-1} \beta_{\varphi(j)}, \quad k > k_0. \tag{4}$$

Let us observe that for $n \in \mathbb{N}$ the iterates $h^{(k)}$ $(k = 1, ..., n)$ are all continuous off the finite set $A_n := \bigcup_{0 \leq j \leq n-1} h^{(-j)}(A)$. Thus if we write $c_k (k \in \mathbb{N})$ for the member of $A_k \cup \{1\}$ closest to the right of x_0, then

$$h^{(k)}(x) = 2^{\sum_{j=0}^{k} \beta_{\varphi(j)}} (x - x_0) + x_k, \quad x \in [x_0, c_k),\ k \in \mathbb{N}. \tag{5}$$

We want to show that the set K of non–negative integers k satisfying $c_{k+1} < c_k$ is finite. Suppose $k \in K$. Then $c_{k+1} = h^{(-k)}(a)$ for some the member a of A. Clearly, $a = a_{\varphi(k)+1}$. Substituting $x = c_{k+1}$ in (5) and solving for c_{k+1}, we obtain $c_{k+1} = (a - x_k)/2^{\sum_{j=0}^{k} \beta_{\varphi(j)}} + x_0$. Denoting $s_0 = n_{k_0} + \sum_{j=0}^{k_0 - 1} \beta_{\varphi(j)}$ and using (4), we get that $\sum_{j=0}^{k} \beta_{\varphi(j)} = s_0 - n_{k+1}, (k \geq k_0)$ and so, if $k \in K$ and $k \geq k_0$ we obtain that

$$\begin{aligned} c_{k+1} &= a 2^{n_{k+1} - s_0} - L_k 2^{n_{k+1} - n_k - s_0}/M' + x_0 = \\ &a_{\varphi(k)+1} 2^{n_{k+1} - s_0} - L_k 2^{-\beta_{\varphi(k)} - s_0}/M' + x_0. \end{aligned} \tag{6}$$

Because the a_j are rational numbers and $n_k \to \infty$ there are only finitely many numbers in $[0,1)$ which can be represented as in the last line of (6). We conclude that the set K is indeed finite, and hence there is a member of $\bigcup_{j \leq 0} h^{(j)}(A) \cup \{1\}$ closest to the right of x_0. Applying the same argument to $h^{(-1)}$, we find a member of $A_\infty \cup \{1\}$ closest to the right of x_0. ■

Theorem 4.2. *Every $h \in \mathcal{C}_1 \cap \mathcal{C}_2$ has a rational rotation number and hence there exists a Borel wandering set for h.*

Proof. If $x_0 = 0$ is a periodic point, the result follows from Proposition 2.5. Otherwise, Lemma 4.1 tells us that $A_\infty \cup \{1\}$ has a smallest positive member. In particular, the orbit of 0 cannot be dense and thus h cannot be conjugate to an irrational rotation. In view of Denjoy's Theorem 2.6, we conclude that the rotation number of h is rational and an appeal to Proposition 2.5 completes the proof. ■

5. Multi–discontinuity case

We prove Theorem 1.2 in this section. The rationale for limiting attention to orbits of break points is valid in a more general setting.

Proposition 5.1. *Every homeomorphism of an open subset of* $\mathbb{R}$ *admits a Borel wandering set.*

Proof. Suppose U is an open subset of $\mathbb{R}$ and $h : U \to U$ is a homeomorphism. Note first that the components of U are intervals and these are permuted by h. Suppose $x \in U$ and write C for its component. If all members of $[x]$ belong to different components of U, then $[x]$ has no cluster points. Otherwise, there is the smallest positive integer n for which $h^{(n)}(x)$ belongs to C. Since $h^{(n)}$ maps C onto itself, the sequence $\{h^{(kn)}(x)\}_{k\in\mathbb{Z}}$ must be monotone, and hence $[x]$ has at most $2n$ cluster points and none of these can lie in $[x]$. Thus no orbit under h has recurrent points and the existence of a Borel wandering set follows from the Glimm–Effros Theorem. ■

We do not know if the cluster point condition of the following Corollary is necessary as well as sufficient for the existence of a Borel wandering set. Theorem 2.4 implies that orbit cluster point sets must in fact be finite when h has at most one discontinuity (not counting 0).

Corollary 5.2. *Suppose h is a piecewise continuous bijection of $[0, 1)$. If the orbit of each discontinuity (including 0) of h has only countably many cluster points, then h admits a Borel wandering set.*

Proof. Write D for the set of discontinuities (including 0) of h, and C for the set of cluster points of its saturation D_∞. The hypothesis assures us that $A := C \cup D$ is countable. By continuity, $C - D_\infty$ is an invariant set for h, so that $A_\infty = C \cup D_\infty$ is a closed countable set. Eliminate redundant members of A to get a countable wandering set F_0 for $h|_{A_\infty}$ and apply Proposition 5.1 to get a Borel wandering set W_0 for $h|_{[0,1)-A_\infty}$. Then $F_0 \cup W_0$ provides a wandering set for h. ■

To apply this idea to Theorem 1.2, we use Lemma 4.1 to show that orbits of the breakpoints involved have countable closures; in fact, the cluster point set turns out to be finite. In showing that the wandering set we get is actually a finite union of intervals, we make a fresh start, working with right–continuous functions and half–open intervals.

So suppose h is a right–continuous bijection of $[0, 1)$ which is piecewise increasing and piecewise linear. We write $\tilde{h}$ for the left–continuous version of h, i.e., $\tilde{h} : (0, 1] \to (0, 1]$ by $\tilde{h}(x) = h(x - 0)$. In the following discussion, A will denote a finite subset of $\mathbb{Q}$ containing the break points of h, and we take $B := A_\infty \cup \{1\}$.

Definition 5.3. Suppose $a < b$ are members of B for which the open interval (a, b) is disjoint from B. Then we call b the *B–successor* of a and refer to the half–open interval $[a, b)$ as a *B–interval.*

Lemma 5.4. *If two B–intervals have a single point in common, then they coincide. If J is a B–interval, then so are all of its images under iterates of h, and all iterates of h are linear on J.*

Proof. The first assertion is clear since the only way for two intervals $[a, b)$ and $[c, d)$ to intersect non–trivially is for one to contain an endpoint of the other in its interior, and this is excluded by the definition of B–interval.

For the second assertion, suppose $J := [a, b)$ is a B–interval. By right continuity, $h(J)$ is the interval $[h(a), \tilde{h}(b))$. Since $B - \{1\}$ is an invariant set, the interior of this interval must be disjoint from B. Of course, if h is continuous at b, then $\tilde{h}(b) = h(b) \in B$. Even in the contrary case however, $\tilde{h}(b) \in h(A) \cup \{1\} \subset B$. The final claim follows since composites of linear maps are linear. ■

Lemma 5.5. *Suppose $J = [a, b)$ is a B–interval. Then J is a wandering set for the restriction of h to J_∞. If a is not periodic, then all orbits of points in J_∞ share a common set of accumulation points. If a has period n, then $h^{(n)}|_{J_\infty}$ is the identity map.*

Proof. Let $J := [a, b)$ be a B–interval. If a is not periodic, then Lemma 5.4 tells us that the intervals $h^{(k)}(J),\ k \in \mathbb{Z}$, are mutually disjoint, whence J provides a wandering set for J_∞. Disjointness of $h^{(k)}(J),\ k \in \mathbb{Z}$ also tells us that their lengths go zero as $|k| \to \infty$, so for any $x \in J_\infty$, the cluster points of $[x]$ are the same as the cluster points of $[a]$.

Suppose, on the other hand, that a has period n. Then Lemma 5.4 tells us that the sets $J, h(J), \dots h^{(n-1)}(J)$ are mutually disjoint, but that $h^{(n)}$ maps J linearly onto itself. This means that $h^{(n)}$ is the identity on J and hence on J_∞. In particular J is a wandering set for $h^{(n)}|_J$ and hence for $h|_{J_\infty}$. ■

We now set up the machinery to apply Lemma 4.1.

Definition 5.6. A cluster point z of B is *regular* if there is an $a \in A$ and integers q, r for which the sequence $\{h^{(jq+r)}(a)\}_{j=1}^{\infty}$ strictly increases to z. A sequence $\{b_n\}$ is said to be a *B–sequence* if b_{n+1} is the B–successor of b_n for each $n \in \mathbb{N}$.

Lemma 5.7. *The set of regular cluster points for h is finite and invariant under $\tilde{h}$.*

Proof. Suppose $z = \lim_{j\to\infty} h^{(jq+r)}(a)$. Since $\tilde{h}$ is left continuous and agrees with h off A, we see that $\tilde{h}(z) = \lim_{j\to\infty} h^{(jq+r+1)}(a)$ so invariance under $\tilde{h}$ is established. Repeating this $|q|$ times, we see that $\tilde{h}^{(q)}(z) = z$, whence the orbit $\mathcal{O}$ of z under $\tilde{h}$ is finite. Further iteration shows that the sequence $\{h^{(jq)}(a)\}_{j=1}^{\infty}$ converges to a member of $\mathcal{O}$. Suppose $z' := \lim_{j\to\infty} h^{(jq'+r')}(a')$ is another cluster point for h. If $a = a'$ and $q, q' > 0$, then the sequence $\{h^{(jqq')}(a)\}_{j=1}^{\infty}$ converges to a point of $\mathcal{O} \cap \mathcal{O}'$, so these orbits coincide. Applying the same argument to h^{-1} shows that $\mathcal{O} = \mathcal{O}'$ when $q, q' < 0$. This shows that each regular cluster point for h falls in one of at most $2|A|$ finite orbits under $\tilde{h}$. ■

Lemma 5.8. *Each B–sequence converges to a regular cluster point for h.*

Proof. Let us note that the finiteness of A provides a natural number ℓ such that $h^{(k)}(a) \notin A$ for all non-periodic $a \in A$ and all integers $|k| \geq \ell$. Another observation we need to point out here, which we have used in the proof of Lemma 4.1, is that $h^{(n)} (n \in \mathbb{N})$ is continuous at all points except possibly the points in $\bigcup_{j=0}^{n-1} h^{(-j)}(A)$ and similarly $h^{(-n)} (n \in \mathbb{N})$ is continuous at all points except possibly the points in $\bigcup_{j=1}^{n} h^{(j)}(A)$. Combining these observations, we conclude that $h^{-(n-\ell)}$ is continuous at $h^{(n)}(a) (n \in \mathbb{N}$, $n \geq \ell$ and $a \in A$ is non-periodic) and similarly $h^{(n-\ell+1)}$ is continuous at $h^{(-n)}(a) (n \in \mathbb{N}$, $n \geq \ell$ and $a \in A$ is non-periodic).

Suppose a, b are non-periodic members of A, and $[h^{(i)}(a), h^{(j)}(b))$ is a B–interval ($i, j \in \mathbb{Z}$, $|j| \geq \ell$). Depending on the sign of j, either $h^{(-j+\ell)}$ or $h^{(-j-\ell)}$ is continuous at the right hand endpoint of this interval. Thus either $h^{(i-j+\ell)}(a)$ or $h^{(i-j-\ell)}(a)$ is a predecessor of a member of the finite set $h^{(\ell)}(A) \cup h^{(-\ell)}(A)$. This means there are only finitely many possibilities for $|i - j|$ (under the above assumptions) and we can choose an upper bound M for its values.

Express the k'th term of our B-sequence as $b_k := h^{(n_k)}(\alpha_k)$, with α_k a member of A. We make several normalizations by postponing the start of the sequence. First, we may assume that none of its terms is periodic. Since $|n_k| \to \infty$, we can also arrange that $|n_k| > \max\{M, \ell\}$ for all k. The preceding paragraph then implies that the powers n_k all have the same sign. For definiteness, we suppose they are all positive; the proof in the opposite case is similar. Finally, we assume that infinitely many terms in the sequence belong to the orbit of a_1.

Choose a positive integer s so that $\alpha_{s+1} = a_1$ and $n_{s+1} > n_1$. Write $q = n_{s+1} - n_1$. Since the powers n_k are strictly positive and greater than ℓ, $h^{(q)}$ is continuous at all points in our sequence. Thus $h^{(q)}$ maps the B–interval $[b_1, b_2)$ onto the B–interval $[b_{s+1}, h^{(q)}(b_2))$. It follows that $b_{s+2} = h^{(q)}(b_2)$, whence an inductive argument yields $b_{s+k} = h^{(q)}(b_k)$ for every $k \in \mathbb{N}$. In particular, for each $j \in \mathbb{N}$, we have $b_{js+1} = h^{(jq+n_1)}(a_1)$ and Lemma 5.8 is established. ■

Lemma 5.9. *Every left–cluster point of B is regular.*

Proof. By Lemma 4.1, we know that each non–periodic member of $B\backslash\{1\}$ has a B–successor. Suppose z were a non–regular left–accumulation point of B. Set

$$F := \{x \in [0,z) : \ x \text{ is a periodic member of } B \text{ or a regular cluster point}\}.$$

Since F is finite, we can choose $b_1 \in B$ so that b_1 is larger than all members of F, but the interval (b_1, z) contains infinitely many members of B. Define $\{b_n\}$ inductively by requiring b_{n+1} to be the B–successor of b_n for each n. The preceding lemma tells us that $\lim b_n$ is a regular cluster point in $(b_1, z]$ which contradicts the choice of b_1. ■

We're now ready to prove Theorem 1.2.

Theorem 5.10. *Suppose h is a bijection of $[0,1)$ whose graph consists of finitely many line segments, with all discontinuities and changes of slope occurring at rationals. If all slopes involved are powers of 2, then h has a wandering set which is a finite union of intervals W_i (some of which may be singleton sets).*

Proof. To begin with, assume h is right–continuous and write A for the set of points (including 0) at which h is either discontinuous or changes slope. Lemma 5.9 tell us that every left–cluster point of A_∞ is regular. Thus from Lemma 5.7, we see that A_∞ has at most finitely many left–cluster points. Applying the same argument to $\tilde{h}$, we conclude that A_∞ has at most finitely many right–cluster points as well, so that the entire cluster point set C of A_∞ is finite. Take $A' := A \cup C$. By continuity, $C - A_\infty$ is invariant under h; moreover, every member of $C - A_\infty$ is a non-interior periodic point for h, and hence must be rational. Thus $A'_\infty := A_\infty \cup C$ is a closed countable set which is invariant under h. Write $B := A'_\infty \cup \{1\}$.

To construct our wandering set, enumerate the finite set A' as $a_1 < \cdots < a_n$. If a_i has an immediate successor b_i in B, take $J_i := [a_i, b_i)$; otherwise take J_i to be the singleton set $\{a_i\}$. Lemma 5.5 tells us that J_i provides a wandering set for the restriction of h to the saturation S_i of J_i. Also, Lemma 5.4 implies that if S_i and S_j share a single point, they will coincide. Since every point of $[0,1)$ which does not belong to C will lie in a unique B–interval, the $\{S_i\}$ in fact form a partition (with repetition) of $[0,1)$. Set $W_i := J_i$ unless $S_i = S_j$ for some $j < i$ in which case we take $W_i := \emptyset$. Then $W := \bigcup_{i=1}^{k} W_i$ provides a wandering set for h.

If h were not right–continuous to start with, we could obtain a right–continuous function h' by permuting the values of h on A. This will only affect orbits of points in A' and the union of those orbits will remain the same. Thus to get from a wandering set W' of h' to a wandering set W for h, it is only necessary to adjust our choice of representatives of finitely many affected orbits. This will not jeopardize the status of W' as a finite union of intervals. ■

A second glance at the application of Lemma 5.5 in the preceding proof yields the dynamical information mentioned in the Introduction.

Corollary 5.11. *For h as is Theorem 5.10, it can be arranged that within a given interval W_i, either all points have the same period, or their orbits share a common cluster point set.*

6. Application to wavelets

In [4], the notion of a wavelet set was introduced, and the question was raised whether the set of all one-dimensional orthonormal wavelets is connected (in the norm topology of $L^2(\mathbb{R})$). One simple way to construct a wavelet is to consider the characteristic function of a (*regularized*) *wavelet set* (i.e. cf. [14], a (Lebesgue) measurable set F satisfying $\mathbb{R} = \cup_{k\in\mathbb{Z}}(F + 2k\pi)$, $\mathbb{R} \setminus \{0\} = \cup_{k\in\mathbb{Z}} 2^k F$ and in both these unions all the sets are mutually disjoint). Given two wavelet sets F and G one can construct two measurable bijection maps $t, d : F \to G$ such that $t(x) - x \in 2\pi\mathbb{Z}$, and $\log_2(d(x)/x) \in \mathbb{Z}$ for all $x \in F$. Let us denote by E the wavelet set $E := [-2\pi, -\pi) \cup [\pi, 2\pi)$. Given a wavelet set F we denote by h_F the measurable bijection on E defined by $t^{-1} \circ d$ where t and d are associated to E and F as above. Let us denote by $\xi : E \to [0, 1)$ the map given by $\xi(x) = x/(2\pi) + 1$, if $x \in [-2\pi, -\pi)$ and $\xi(x) = x/(2\pi)$, if $x \in [\pi, 2\pi)$. Conjugation of h_F by the map ξ, $\widetilde{h}_F := \xi \circ h_F \circ \xi^{-1}$, will be called a *wavelet induced map.* In [14], it was proved that E and F can be connected by a continuous path of wavelets sets if the function $\widetilde{h}_F$ admits a measurable wandering set (in the sense of Definition 2.1). Combining Theorem 5.10 with Theorem 2 of [14], we obtain the following proposition.

Proposition 6.1. *Suppose F is a wavelet set which is a finite union of intervals whose endpoints are rational multiples of π. Then F can be connected with E by a continuous path of wavelet sets which are countable unions of intervals contained in the union of E and F.*

Proof. The only thing we need to check is that $\widetilde{h}_F$ has the form described in Theorem 5.10. Since F is a finite union of intervals whose endpoints are rational multiples of π there are finitely many pieces (intervals) in the definition of $d : E \to F$ and the endpoints for these intervals are rational multiples of π. Since F is a wavelet, we may assume that $(-\epsilon, \epsilon) \cap F = \emptyset$ for some $\epsilon > 0$. This implies that there are finitely many intervals in the definition of $t : E \to F$ too and the endpoints of these intervals are also rational multiples of π. Then $h_F = t^{-1} \circ d : E \to E$ is piecewise linear and clearly it has the form $h_F(x) = 2^k x + r\pi$ (for some $k \in \mathbb{Z}$ and $r \in \mathbb{Q}$) on finitely many intervals. This gives the desired form for the map $\widetilde{h}_F : [0, 1) \to [0, 1)$. ■

In [9], four-interval wavelet sets were classified and it is clear that all discrete families of wavelet sets in [9] are as in the hypothesis of Proposition 6.1. So, one can apply Proposition 6.1 to all these discrete families of four interval wavelet sets and conclude that they can be connected with E by special paths of wavelet sets. One can check that for some other one-parameter families of wavelets in [4], [9], or [17], the corresponding wavelet induced maps admit wandering sets too. In fact, these wavelet set families indexed over some interval (or some of them even over a Cartesian product of two intervals as described in [9]) depend continuously upon these parameters. Since one can always choose some rational multiples of π for these parameters, it follows that we can apply our result to at least one element in these families. Therefore, all wavelet sets which are finite unions of intervals described in [4], [9], or [17] can be connected with E virtue of Proposition 6.1.

Finally, we consider a concrete example of a four interval wavelet set from [9] to illustrate our techniques. Let $F = \left[-\frac{640}{21}\pi, -\frac{448}{15}\pi\right) \cup \left[-\frac{56}{15}\pi, -\frac{8}{3}\pi\right) \cup \left[-\frac{2}{3}\pi, -\frac{10}{21}\pi\right) \cup \left[\frac{2}{15}\pi, \frac{4}{15}\pi\right)$. One can easily check that F is a wavelet set using Lemma 1.1 in [14] by computing the maps d and t for this case. Then the corresponding wavelet induced map is given by

$$\widetilde{h}_F(s) = \begin{cases} \frac{s+3}{4}, \quad s \in \left[0, \frac{1}{21}\right), \\ 16s, \quad s \in \left[\frac{1}{21}, \frac{1}{16}\right), \\ 16s-1, \quad s \in \left[\frac{1}{16}, \frac{1}{15}\right), \\ 2s, \quad s \in \left[\frac{1}{15}, \frac{1}{3}\right), \\ \frac{s+1}{2}, \quad s \in \left[\frac{1}{3}, \frac{1}{2}\right), \\ \frac{s}{4}, \quad s \in \left[\frac{1}{2}, \frac{8}{15}\right), \\ \frac{s}{8}, \quad s \in \left[\frac{8}{15}, 1\right). \end{cases}$$

To apply Theorem 5.10 to the function $f := \tilde{h}_F$, start with the set $A := \left\{0, \frac{1}{21}, \frac{1}{16}, \frac{1}{15}, \frac{1}{3}, \frac{1}{2}, \frac{8}{15}\right\}$ of break points. Using Maple software, one finds the set of cluster points of the forward orbit of 0 to be $\mathcal{O} := \left\{\frac{2}{3}, \frac{1}{12}, \frac{1}{6}, \frac{1}{3}\right\}$; these are approached from the right. The backward orbit of 0 converges to $\frac{1}{15}$ from the left. A brief examination of the other points of A shows these are the only cluster points of A_∞. As predicted by Lemmas 5.7 and 5.9, $\tilde{f}(1/15) = 1/15$; dually, $\mathcal{O}$ is invariant under f. In this example, A_∞ contains all of its cluster points, so we can just take $A' := A$. Set $B := A_\infty \cup \{1\}$ as usual.

Next, we find the B–successors of the members of A. These are $\frac{1}{21}$ for 0, $\frac{1}{16}$ for $\frac{1}{21}$, $\frac{1}{12}$ for $\frac{1}{15}$, none for $\frac{1}{3}$, and $\frac{8}{15}$ for $\frac{1}{2}$. Remaining members of A are redundant since they belong to orbits of points already considered. Since $\frac{1}{3}$ has no successor, we include it in our wandering set, arriving at $W := \left[0, \dfrac{1}{16}\right) \cup \left[\dfrac{1}{15}, \dfrac{1}{12}\right) \cup \left\{\dfrac{1}{3}\right\} \cup \left[\dfrac{1}{2}, \dfrac{8}{15}\right)$.

The linearity of iterates of f on B–intervals gives us global dynamical information. For example, since $\frac{1}{15}$ has period 4, we conclude that $f^{(4)}$ must be the identity on the saturation of the B–interval $\left[\dfrac{1}{15}, \dfrac{1}{12}\right)$, i.e., on the set $E_4 := \left[\dfrac{1}{15}, \dfrac{1}{12}\right) \cup \left[\dfrac{2}{15}, \dfrac{1}{6}\right) \cup \left[\dfrac{4}{15}, \dfrac{1}{3}\right) \cup \left[\dfrac{8}{15}, \dfrac{2}{3}\right)$; for similar reasons $f^{(3)}$ is the identity on $E_3 := \left[\dfrac{1}{8}, \dfrac{2}{15}\right) \cup \left[\dfrac{1}{4}, \dfrac{4}{15}\right) \cup \left[\dfrac{1}{2}, \dfrac{8}{15}\right)$. On the other hand, since the intervals $f^{(n)}([0, \dfrac{1}{16})), n \in \mathbb{Z}$, are disjoint, their lengths must go to zero as $|n| \to \infty$. It follows that every point outside $E_3 \cup E_4 \cup \mathcal{O}$ will behave like 0: forward orbits must asymptotically approach $\mathcal{O}$ from the right, while backward orbits will converge to $\dfrac{1}{15}$ from the left.

7. A SPECIAL 3-SLOPE EXAMPLE

Let φ be a homeomorphism of $[0,1)$. As mentioned in Section 2, translations modulo one of φ can be regarded as homeomorphisms of the unit circle. In [16], Khanin and Vul show that if φ has a single "weak discontinuity" (i.e. change of slope) on the unit circle, then almost all translates of φ have rational rotation numbers.

In this section, we study the translates $g_t := R_t \circ \varphi$, $t \in [0,1)$, of the function φ defined by

$$\varphi(x) = \begin{cases} 2x, & x \in [0, 1/3), \\ (x+1)/2, & x \in [1/3, 1) \end{cases} .$$

where $R_t(x) = x + t \ (mod\ 1), \quad x \in [0,1)$. Our function φ has two weak discontinuities and so it is not covered by the work in [16], but as predicted by Theorem 1.2, all its rational translates, g_t ($t \in \mathbb{Q}$), have rational rotation numbers. The next theorem also provides a numerical algorithm to compute these rotation numbers.

It turns out that the family of isomorphisms $\{g_t\}$ is strongly related to the family $\{h_t\}$ defined by

$$h_t(x) = \begin{cases} x + t, & x \in [0, (1-t)), \\ 2(x - 1 + t), & x \in [1-t, 1-2t/3), \\ x/2 + t - 1/2, & x \in [1-2t/3, 1) \end{cases} \quad t \in [0,1).$$

Let us observe that g_t , $h_t \in \mathcal{C}_2$ for every $t \in \mathbb{Q}$. For every $t \in (0,1)$ we denote by $r_h(t)$ [resp. $r_g(t)$] the rotation numbers for h_t [resp. g_t] as defined in (1) and as a convention here and for unity let us set $r_h(0) = 0$ [resp. $r_g(0) = 0$]. We obtain the following theorem.

Theorem 7.1. *The rotation number maps $t \to r_h(t),\ t \to r_g(t)$ satisfy the functional equations:*

$$r_h\left(\frac{1}{n-t}\right) = 1/(n - r_g(t)), \quad t \in [0,1), \quad n \in \mathbb{N} \setminus \{1\}. \tag{7}$$

$$r_g\left(\frac{1}{(3-t)2^n - 3}\right) = 1/(2n - r_h(t)), \quad t \in [0,1), \quad n \in \mathbb{N}, \tag{8}$$

$$r_g\left(\frac{1}{(4-t)2^n - 3}\right) = 1/(2n + 1 - r_h(t)), \quad t \in [0,1), \quad n \in \mathbb{N}. \tag{9}$$

Proof. Fix $n \in \mathbb{N}\backslash\{1\}$. If $t = 0$ then the forward orbit of 0 under $h_{\frac{1}{n}}$ is just $\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}\}$. By (1) we get $r_h(\frac{1}{n}) = \frac{1}{n}$ which shows, by the convention we made, that (7) is true for this case. Tracing the orbit of 0 under $g_{\frac{1}{3(2^n)-3}}$ [respectively $g_{\frac{1}{4(2^n)-3}}$] will show the validity of (8) and (9) for the case $t = 0$.

In order to establish (7) for $t \in (0,1)$, we compute the first return map ([20]) of h_s to the interval $[0,s)$ for $s = \frac{1}{n-t}$, and show that this is precisely, up to scaling, the map $\widetilde{g}_t = \varphi \circ R_t$. An arbitrary point in $[0,s)$ can be written as an xs with $x \in [0,1)$. Hence we need to trace the forward orbit of xs under the action of h_s. Let us write $x_k = h_s^{(k)}(xs)$, $k \in \mathbb{N} \cup \{0\}$ and denote $I_1 := [0, 1-s)$ and $I_2 := [1-s, 1)$.

Case I. We assume first that $x+t < 1$. Then a simple computation shows that $x_0,\ x_1, ...,\ x_{n-2} \in I_1$ and $x_{n-1} \in I_2$. Moreover, $x_1, ..., x_{n-1} \notin [0,s)$. Now, the inequality $x_{n-1} < 1 - 2s/3$ is equivalent to $x+t < 1/3$ and if the former inequality is true then $x_n = \varphi(x+t)/(n-t) = \widetilde{g}_t(x)s \in [0,s) \subset I_1$. In case $x + t \geq 1/3$ another computation shows that still $x_n = \widetilde{g}_t(x)s \in [0,s)$.

Case II. Let us consider that $x + t \geq 1$. In this case, the first element in I_2 is x_{n-2} and a similar calculation gives $x_{n-1} = \varphi(x+t-1)/(n-t) = \widetilde{g}_t(x)s$ which is the first term of $\{x_k\}$ to return to $[0,s)$.

This analysis implies that the first return map of h_s to the interval $[0,s)$ is $\widetilde{g}_t$ up to scaling. To compute the rotation number for h_s let us use (1). We clearly have the first $n - 1 - \chi_{[1-t,1)}(x)$ terms of $\{x_k\}$ in I_1 and then the next one in I_2; then the following next $n-1-\chi_{[1-t,1)}(\widetilde{g}_t(x))$ terms in I_1 and the next one in I_2 and so on. Hence, since $\widetilde{g}_t = R_t^{-1} \circ g_t \circ R_t$, we have

$$r_h(s) = \lim_{p\to\infty} \frac{p}{pn - \sum_{j=0}^{p} \chi_{[1-t,1)}(\widetilde{g}_t^{(j)}(x))} = \frac{1}{n - r_{\widetilde{g}}(t)} = \frac{1}{n - r_g(t)},$$

which establishes (7).

To prove (8) for $t \in (0,1)$ we proceed in a similar way by computing the first return map of g_s to the interval $[0,s)$, where $s = \frac{1}{(3-t)2^n-3}$. So, let us fix an $x \in [0,1)$ and as before we denote by $\{x_k\}$ the orbit of xs under g_s.

Case I. We assume that $x < 1-t$. Then, one can compute $x_0,\ x_1, ...,\ x_{n-1}$ and see that these are all in the interval $[0,1/3)$ and that $x_n = [2^n(x+1)-1]/[(3-t)2^n-3] \geq 1/3$. Also, $x_{n+j} = [(3-t)2^n + (x+t-2)2^{n-j}-1]/[(3-t)2^n-3]$, $j = 0, ...,\ n-1$, which implies that x_{2n-1} is the first term of $\{x_k\}$ with the property $\varphi(x_{2n-1}) + s \geq 1$ and so $x_{2n} = (x+t)s = h_t(x)s$ is the first term to return to $[0,s)$.

Case II. Let us consider $x \in [1-t, 1-2t/3)$. Again, it is easy to check that $x_0,\ x_1, ..., x_{n-1} \in [0,1/3)$, $x_n, ..., x_{2n-2} \in [1/3,1)$, and x_{2n-2} is the first term of $\{x_k\}$ with the property $\varphi(x_{2n-2})+s \geq 1$ and so $x_{2n-1} = 2(x+t-1)s = h_t(x)s$ is the first term to return to $[0,s)$.

Case III. If $x \in [1-2t/3, 1)$, then the first term of $\{x_k\}$ in the interval $[1/3,1)$ is $x_{n-1} = [2^{n-1}(x+1)-1]/[(3-t)2^n-3]$. Hence, a similar computation shows that $x_{n-1+j} = [(3-t)2^n + (x+2t-5)2^{n-1-j}-1]/[(3-t)2^n-3]$, $j = 0, ..., n-1$ which implies that x_{2n-2} is the first term in $\{x_k\}$ with the propety $\varphi(x_{2n-2}) + s \geq 1$ and so $x_{2n-1} = [(x+2t-1)/2]/[(3-t)2^n-3] = h_t(x)s$ is the return in this case.

This analysis says that the first return map of g_s to the interval $[0,s)$ is, up to scaling, exactly h_t. To compute the rotation number of g_s we again use (1) and a similar argument as above to get

$$r_g(s) = \lim_{p\to\infty} \frac{p}{2pn - \sum_{j=0}^{j=p} \chi_{[1-t,1)}(h_t^{(j)}(x))} = 1/(2n - r_h(t)).$$

The equality (9) can be proved in a similar way. ■

Corollary 7.2. *If $t \in \mathbb{Q} \cap [0,1)$, then $r_h(t),\ r_g(t) \in \mathbb{Q}$ and one can use (7)—(9) to compute numerically these rotation numbers for any given rational number t. In particular, there exist Borel wandering sets for h_t and g_t whenever $t \in \mathbb{Q}$.*

Proof. Let us consider the sets $Q_g := \{t;\quad r_g(t) \in \mathbb{Q}\}$ and $Q_h := \{t;\quad r_h(t) \in \mathbb{Q}\}$. Using (7)—(9) we obtain that if $t \in Q_g$ then $\frac{1}{n-t} \in Q_h$ and hence $\gamma(t,m,n) := [(3-\frac{1}{n-t})2^m - 3]^{-1} \in Q_g$ and $\gamma'(t,m,n) := [(4-\frac{1}{n-t})2^m-3]^{-1} \in Q_g (n,m \in \mathbb{N},\ n \geq 2)$.

Conversely, if for some $n,m \in \mathbb{N} (n \geq 2)$ and $t \in [0,1)$ we have $\gamma(t,m,n) = s \in Q_g$ or $\gamma'(t,m,n) = s \in Q_g$, then $t \in Q_g$. Indeed, if s is of the form $\frac{1}{4\cdot 2^m-3}$ [resp. $\frac{1}{3\cdot 2^m - 3}$]$(m \in \mathbb{N})$ or $s = 0$, then by (8) and (9) we see that $s \in Q_g$. If $s \neq 0$ and s is not of the above form, there exist uniquely determined integers $m,n \in \mathbb{N}$, $n \geq 2$, and $t \in [0,1)$ such that $\gamma(t,m,n) = s$ or $\gamma'(t,m,n) = s$ with the condition $\frac{1}{n-t} \in [0,1)$. This fact is basically consequence of the fact that

$$(0,1) = \bigcup_{m\in\mathbb{N}} \left(\left[\frac{1}{4\cdot 2^m - 3}, \frac{1}{3\cdot 2^m - 3} \right) \cup \left[\frac{1}{3\cdot 2^m-3}, \frac{1}{2\cdot 2^m - 3} \right) \right) = \bigcup_{n\in\mathbb{N}, n\geq 2} \left[\frac{1}{n}, \frac{1}{n-1} \right).$$

We denote the unique $t \in [0,1)$ satisfying the above conditions by $\gamma^{-1}(s)$ for all s as described above. It is clear that if s is rational then $\gamma^{-1}(s)$ is rational. We notice that for $s \geq 3/4$, one can check that h_s has a fixed point and therefore $r_h(s) = 1$ for every $s \geq 3/4$. Using (8), we obtain that

$$r_g(s) = 1, \quad s \geq 2/3. \tag{10}$$

To finish the proof, it suffices to show that if $t = p/q$ (reduced form) then $\gamma^{-1}(t) = p'/q'$ (reduced form) with $q' < q$ unless $p'/q' \geq 2/3$. Indeed, for $t = p/q < 2/3$ we construct the sequence of the first rational numbers $t, \gamma^{-1}(t), \gamma^{-1}(\gamma^{-1}(t)), ...$, which are less then $2/3$. This sequence of fractions (written in reduced form) would be finite since the denominators of these fractions would form a decreasing sequence of nonnegative integers. This implies that if one starts with a rational number t, one can compute the rotation number $r_g(t)$, using the above sequence, identities (7)–(9) and (10).

For instance, if $t = 10/231$, then $\gamma^{-1}(t) = 38/59$, $\gamma^{-2}(t) = 34/55$, $\gamma^{-3}(t) = 26/47$, $\gamma^{-4}(t) = 10/31$, and $\gamma^{-5}(t) = 18/19 > 2/3$. Then going backwards one gets $r_g(18/19) = 1$ and then from (7), $r_h(19/20) = 1$. Alternately, from (8), $r_g(10/31) = 1/2$, and so on. Finally, we one gets $r_g(10/231) = 9/55$.

In order to prove our claim we show equivalently that if $\frac{p}{q} \in [0, 2/3)$ (reduced form), then for every $n, m \in \mathbb{N}(n \geq 2)$, we get $\gamma(\frac{p}{q}, m, n) = \frac{p'}{q'}$ [resp. $\gamma'(\frac{p}{q}, m, n) = \frac{p'}{q'}$], where $(p', q') = 1$ and $q' > q$. Let us consider the case of γ first. Clearly, we have

$$\frac{p'}{q'} = \frac{(nq-p)}{[3(nq-p)-q]2^m - 3(nq-p)}. \tag{11}$$

Suppose that the prime number l divides both the numerator and denominator of the right-hand-side fraction of (11). Then if $l \neq 2$ it follows that l divides q and p which is not possible. This implies that the right-hand-side fraction of (11) cannot be simplified except by an integer 2^k ($k \in \mathbb{N} \cup \{0\}$).

Case I. We assume first that $k = 0$. Then $q' = [3(nq-p)-q]2^m - 3(nq-p) \geq [3(nq-p)-q]2 - 3(nq-p) = (3n-2)q - 3p > q$.

Case II. Next, we may assume that $k \geq 1$ and let us write $nq - p = 2^k l$, $l \in \mathbb{N}$, $k \leq m$. Hence $q' = [(3n-1)q - 3p]2^{m-k} - 3l \geq (3n-1)q - 3p - 3l = 3(nq - p - 2l)/2 + (3n/2 - 1)q - 3p/2 \geq (3n/2-1)q - 3p/2 > q$ since the last inequality is equivalent to $(3n-4)/3 > p/q$ which is true under our hypothesis that $p/q < 2/3$.

With similar arguments one can show that still $q' > q$ if $\gamma'(\frac{p}{q}, m, n) = \frac{p'}{q'}$(reduced form). ■

Corollary 7.3. *For $(m, n) \in \mathbb{N} \times (\mathbb{N}\backslash\{1\})\backslash\{(1, 2)\}$, each of the equations*

$$x^2 - \left(n - \frac{1}{3}\right)x + \frac{n}{3(2^m - 1)} = 0 \quad \text{and} \quad x^2 - nx + \frac{n}{2m} = 0, \tag{12}$$

has only one solution in the interval $[0, 1)$ which we denote by t_1 and t_2 respectively. Then $r_g(t_1) = t_2$. (For instance, if $(m, n) = (2, 2)$, we get $r_g((5 - \sqrt{17})/6) = 1 - 1/\sqrt{2}$.)

Proof. It is easy to see that each of the quadratic equations in (12) has a unique solution in $[0,1)$. The equation satisfied by t_1 can be written as $t_1 = \gamma(t_1, m, n)$ and so, using (7) and (8) it follows that $r_g(t_1)$ satisfies the equation

$$x = \frac{1}{2m - \frac{1}{n-x}}$$

which has the unique solution $t_2 \in [0,1)$. ∎

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