ON RANK TWO LINEAR TRANSFORMATIONS AND REFLEXIVITY

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Abstract. We study operator algebras generated by commuting families of nilpotents. In order for such an algebra $\mathcal{A}$ to be reflexive, it is necessary that each ideal generated by a rank two member of $\mathcal{A}$ be one–dimensional. When the underlying space is a finite–dimensional Hilbert space and the nilpotents in question doubly commute in the sense that they commute with each other's adjoints, the condition is also sufficient.

Doubly commuting families of nilpotents admit simultaneous Jordan Canonical Forms and reflexivity of $\mathcal{A}$ can also be characterized in terms of Jordan block sizes. In particular, our results generalize work of J. Deddens and P. Fillmore on singly–generated operator algebras.

1. Introduction. Most linear transformations in this paper act on finite–dimensional complex vector spaces; the terms operator and linear transformation are used interchangeably. An operator algebra $\mathcal{A}$ is said to be *reflexive* if it is determined by its invariant subspaces in the sense that any operator leaving each member of $Lat\mathcal{A}$ invariant must belong to $\mathcal{A}$. Each reflexive algebra includes an appropriate identity operator, so all operator algebras considered below will be unital. An individual operator is said to be *reflexive* if it generates a reflexive algebra.

In their paper [4], J. Deddens and P. Fillmore characterized reflexive operators in terms of their Jordan Canonical Forms. The old trick of embedding arbitrary linear spaces of operators in commutative operator algebras, not to mention lack of a suitable notion of canonical form, seem to rule out unrestricted generalization of the Deddens–Fillmore result to multi–generated algebras.

The first step in the Deddens–Fillmore analysis is reduction to the nilpotent case. A similar procedure is possible in the multi–operator setting, but we have chosen to postpone the argument to the end of the paper (Theorem 7.4), concentrating on commuting families of nilpotents in the main discussion.

A commuting N–tuple of Hilbert space operators is said to *doubly commute* if the operators commute with each other's adjoints. Previous work [10],[11] of the second author suggested that this might provide a favorable setting for generalization of

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the Deddens–Fillmore result. Examination of key examples led to a notion of simultaneous Jordan Forms and the framing of an appropriate conjecture concerning the relationship between reflexivity and block sizes.

Absence of a natural total order on pairs of integers promised interesting applications of the resulting theorem, but also foreshadowed an awkward proof. In seeking a more conceptual approach, we reexamined the Deddens–Fillmore condition, eventually discovering the connection between block sizes and operators of rank two. The following theorem should be regarded as the main result of the paper.

Theorem A. *Suppose $\mathcal{A}$ is an operator algebra generated by a commuting family of nilpotents. Then in order for $\mathcal{A}$ to be reflexive it is necessary that each rank two member of $\mathcal{A}$ generate a one–dimensional ideal. If the underlying vector space is a finite–dimensional Hilbert space and the generators for $\mathcal{A}$ commute with each other's adjoints, then this condition is also sufficient.*

It is possible to apply Theorem A directly to concrete examples. Thus one can see that the algebra

$$(*) \qquad \left\{ \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ & \alpha & & \gamma \\ & & \alpha & \beta \\ & & & \alpha \end{pmatrix} \oplus \begin{pmatrix} \alpha & \beta & \epsilon \\ & \alpha & \beta \\ & & \alpha \end{pmatrix} \oplus \begin{pmatrix} \alpha & \gamma \\ & \alpha \end{pmatrix} \ : \ \alpha, \beta, \gamma, \delta, \epsilon, \in \mathbb{C} \right\}$$

is reflexive, but the algebra

$$(**) \qquad \left\{ \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ & \alpha & & \gamma \\ & & \alpha & \beta \\ & & & \alpha \end{pmatrix} \oplus \begin{pmatrix} \alpha & \beta & \epsilon \\ & \alpha & \beta \\ & & \alpha \end{pmatrix} \oplus \begin{pmatrix} \alpha & \beta \\ & \alpha \end{pmatrix} \ : \ \alpha, \beta, \gamma, \delta, \epsilon, \in \mathbb{C} \right\}$$

is not reflexive (missing entries are assumed to be zero). In Section 6, we will study such examples in more detail in order to illustrate the full strength of Theorem A. In the process, we present the original and more easily applicable version of the theorem. In order to simplify the notation, we concentrate on doubly commuting pairs (a, b) of nilpotents. For our simultaneous Jordan form, we consider direct sum decompositions of (a, b). As in the single operator case, the sizes of these direct summands provide a complete set of invariants for these pairs. We store this information in a finite "Jordan sequence" $(m_1, n_1), \dots, , (n_k, m_k)$; precise definitions are given in Section 6.

This leads to the following concrete version of Theorem A for pairs of operators. It reduces to the Deddens–Fillmore result when $b = 0$.

Theorem B. *Suppose (a, b) is a doubly commuting pair of nilpotents acting on a finite–dimensional Hilbert space with Jordan sequence $(m_1, n_1), \dots, , (m_k, n_k)$. Then the algebra $\mathcal{A}(a, b)$ generated by a, b is reflexive if and only if for each index i,*

(1) *if $m_i \geq 2$, we can find $j \neq i$ with $m_j \geq m_i - 1$ and $n_j \geq n_i$, and*
(2) *if $n_i \geq 2$, we can find $j \neq i$ with $n_j \geq n_i - 1$ and $m_j \geq m_i$.*

The structure of the balance of the paper is as follows. In Section 2, we motivate Theorem A by "working backwards" in the single operator case from Jordan block sizes to ideals generated by operators of rank two. In Section 3, we establish necessity of the condition in Theorem A in the general context of operator algebras generated by commuting nilpotents—this is the one portion of the paper where finite–dimensionality is not needed.

The main result of Section 4 provides necessary and sufficient conditions for reflexivity of *sub*direct sums of operator algebras having separating vectors. The relevance of the topic to establishing sufficiency in Theorem A lies in the gap between algebras generated by direct sums of operators and full direct sums of algebras generated by individual direct summands.

In Section 5, we introduce our multi–operator version of simple Jordan block for doubly–commuting N–tuples and prove sufficiency in Theorem A. An expanded version of Theorem B is established in Section 6. This section also contains applications of the main results to tensor product algebras and illustrates the increased complexity of the multi–operator setting.

Since reflexivity is invariant under similarity, it is natural to seek a purely algebraic characterization of those N–tuples which are simultaneously similar to doubly commuting N–tuples. This turns out to be that the members of the N–tuple belong to mutually commuting semisimple algebras. The resulting generalization of Theorem A is presented in the final section of the paper. We also consider the non–nilpotent case in this section.

2. Single operator case. All vector spaces studied in this paper are complex, and with the exception of Section 3, finite–dimensional. No norms or topologies are needed, and the terms *linear transformation* and *operator* are used interchangeably. While the hypothesis of double–commutativity in Theorem A implicitly involves inner products, these will not be needed until Section 5. A linear manifold of operators is called an *operator space* ; an *operator algebra* is an operator space which is closed under composition and includes the identity operator I.

Let $\mathcal{A}$ be an operator subalgebra of $L(V)$. Then $Lat\mathcal{A}$ denotes the collection of subspaces of V left invariant by each member of $\mathcal{A}$, while $AlgLat\mathcal{A}$ stands for the collection of operators which leave each member of $Lat\mathcal{A}$ invariant. $\mathcal{A}$ is said to be *reflexive* if it is determined by its lattice of invariant subspaces in the sense that $\mathcal{A} = AlgLat\mathcal{A}$. An individual operator is called *reflexive* if the operator algebra it generates is reflexive. Given a set $\mathcal{F} \subset L(V)$, we write $\mathcal{A}(\mathcal{F})$ for the algebra generated by $\mathcal{F}$.

As promised in the Introduction, the present section is motivational, formal proofs being omitted as these results are consequences of Theorem 6.2. We begin by recalling the Deddens–Fillmore result. A nilpotent operator is said to be *simple* if its Jordan form consists of a single block.

Theorem 2.1. *Let a_1, a_2 be nilpotents of orders $m_1 \geq m_2$ respectively.*

(1) *If m_1 and m_2 differ by at most one, then $a_1 \oplus a_2$ is reflexive.*
(2) *If a_1 is simple and m_1 and m_2 differ by more than one, then $a_1 \oplus a_2$ is not reflexive.*

Let a be nilpotent. Apply the Jordan Canonical Form Theorem to write $a =$

$\oplus_{i=1}^k a_i$, where the direct summands are simple. It is convenient to assume that $k \geq 2$; this can always be accomplished by including a direct summand acting on a zero–dimensional space (considered to have order zero) in the decomposition. Theorem 2.1 immediately tells us whether a is reflexive in terms of the orders $m_1, \ldots, m_k$ (not necessarily monotone) of the blocks $a_1, \ldots, a_k$. Indeed it is clear that a is reflexive if and only if the following condition holds.

(0) The orders of two largest blocks differ by at most one.

It will be useful to have several alternate versions of (0). We aim for characterizations which generalize naturally to the multi–operator case. One of the difficulties we face is that there is no natural ordering on pairs of natural numbers.

The following immediate restatement of (0) is the single–operator version of the condition in Theorem B.

(1) For each i, if $m_i \geq 2$ then there is some $j \neq i$ with $m_j \geq m_i - 1$.

Next, suppose (0) holds, write m for the order of a, and consider the rank of a^{m-2}. Each block of order m in the Jordan decomposition of a contributes 2 to this rank, while each block of order $m-1$ contributes 1. Thus we must have rank $(a^{m-2}) > 2$. A fortiori, the ranks of smaller powers of a must also exceed 2. Moreover, the rank of any polynomial in a is determined by its term of lowest degree. In other words, the only members of $\mathcal{A}(a)$ which are allowed to have rank 2 are scalar multiples of a^{m-1}. Observing that no block of a^{m-1} can have rank two, we are led to the following version of (0).

(2) If $c = \oplus_{i=1}^k c_i \in \mathcal{A}(a)$ has rank 2, then $c_i \neq 0$ for two values of i.

The following reformulation has the advantage of not depending on the decomposition.

(3) If $c \in \mathcal{A}(a)$ has rank 2, then $c\mathcal{A}(a)$ is one–dimensional.

Complete proofs of the equivalence of (0),(1),(2),(3) will follow from the multi–operator case considered in Section 6.

3. Necessity. In this section, we prove necessity in Theorem A and present some related examples. There is no restriction on the dimension of the underlying vector space. Actually, there are two notions of reflexivity in this setting—in addition to the *algebraic* version featured in this paper, there is a *topological* version in which the underlying vector space is equipped with a norm and *Lat* and *Alg* only concern *closed* subspaces and *bounded* operators. We omit further details since the results (and proofs) of this section are valid under both interpretations.

On the other hand, it is crucial to recall an extension of the reflexivity concept originally due to A. I. Loginov and V. I. Sulman [8]. The *reflexive closure* of an operator space $\mathcal{S} \subset L(V)$ is defined by $Ref\, \mathcal{S} = \{a \in L(V) : a\, x \in \mathcal{S}\, x \text{ for all } x \in V\}$. The space S is called *reflexive* if $\mathcal{S} = Ref\, \mathcal{S}$. When $\mathcal{S}$ is an algebra with identity, then $Ref \mathcal{S} = Alg Lat \mathcal{S}$, and the current notion of reflexivity reduces to the classical concept.

We also recall two basic properties of nilpotent operators.

Lemma 3.1. *Let a be nilpotent and suppose c is a non–zero operator of finite rank commuting with a. Then rank$(a\,c) <$ rank(c).*

Proof. We know that range$(c\,a) \subseteq$ range(c). If this inclusion was not proper, we would have range$(c\,a^n) =$ range(c), for all n. But this is ruled out by the nilpotence of a.

Given x in V and ϕ in the dual space V', we write $x \cdot \phi$ for the operator on V defined by $(x \cdot \phi)z = \phi(z)x$, for $z \in V$. This is the zero operator if $x = 0$ or $\phi = 0$. Otherwise $x \cdot \phi$ has rank one, and it is clear that x spans its range and it has the same kernel as ϕ. Moreover every rank one operator has this form.

Lemma 3.2. *If b is a nilpotent commuting with the rank one operator $x \cdot \phi$ then $b\,x = 0$.*

Proof. Applying Lemma 3.1, we conclude that $(b\,x) \cdot \phi = b(x \cdot \phi) = 0$ and the conclusion follows since $\phi \neq 0$.

Consider the simplest non–reflexive algebra, $\left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix} : \lambda, \mu \in \mathbb{C} \right\}$. Note that the identity operator, which is of rank 2, does not generate a one–dimensional ideal. The following proposition (necessity in Theorem A) shows that this is enough to prevent reflexivity.

Proposition 3.3. *Suppose $\mathcal{A}$ is a reflexive algebra generated by commuting nilpotents. Then each $c \in \mathcal{A}$ of rank two must generate a one–dimensional ideal.*

Proof. Assume that $c \in \mathcal{A}$ has rank 2, but fails to generate a one–dimensional ideal. Thus there is some $b \in \mathcal{A}$, such that $b\,c$ is independent of c. Subtracting a multiple of the identity from b if necessary, we can assume that b is a nilpotent whence rank$(c\,b) = 1$ by Lemma 3.1. Choose $x \in V, \phi \in V'$ with $c\,b = x \cdot \phi$. Write $c = x \cdot \psi + w \cdot \xi$ for appropriate w, ψ, ξ. Since b commutes with $c\,b = x \cdot \phi$, we have $b\,x = 0$ by Lemma 3.2. Hence $b\,c = b\,w \cdot \xi = x \cdot \phi$. This forces ξ to be a scalar multiple of ϕ, so changing w if necessary, we can write $c = x \cdot \psi + w \cdot \phi$.

We will complete the proof by showing that the rank one operator $x \cdot \psi$ belongs to $Ref(\mathcal{A})$, but not to $\mathcal{A}$. For the first assertion, note that for $y \notin \ker \phi$, we have $(x \cdot \psi)y = \frac{\psi(y)}{\phi(y)} b\,c\,y$, while for $y \in \ker \phi$, we get $(x \cdot \psi)y = c\,y$.

Suppose, on the other hand, that $x \cdot \psi \in \mathcal{A}$. Then b commutes with $x \cdot \psi$. Moreover, since we already know that b commutes with c, we also learn that b commutes with $w \cdot \phi$. But then Lemma 3.2 yields $b\,x = b\,w = 0$ which leads to the contradiction $b\,c = 0$.

Example 3.4. The nilpotent matrices $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ generate the full algebra of two by two matrices. Since this is a reflexive algebra whose rank two member $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ does not generate a one–dimensional ideal, it is not possible to drop the commutativity hypothesis in Proposition 3.3.

Example 3.5. To see that the condition in Proposition 3.3 does not guarantee reflexivity, fix a subspace $\mathcal{S}$ of $L(V)$ with dim$V \geq 2$ and take $\mathcal{A} = \left\{ \begin{pmatrix} \lambda I & a \\ 0 & \lambda I \end{pmatrix} : a \in \mathcal{S} \right\}$. Then the algebra $\mathcal{A}$ is commutative and each of its rank two members generates a one–dimensional ideal. On the other hand, if $\mathcal{S}$ is not reflexive, for example the algebra mentioned before Proposition 3.3, then neither is $\mathcal{A}$. See for example [2],[3].

For a more striking example, apply [2,Proposition 3.7] to get an operator space $\mathcal{T}$ which is not 3–reflexive and take $\mathcal{S}$ to be a three–fold copy $\mathcal{T}$. Then the resulting algebra $\mathcal{A}$ fails to be reflexive even though it has *no* rank two members.

4. Reflexivity of subdirect sums. From now on, all underlying vector spaces are assumed to be finite–dimensional. It is easy to see that a full direct sum $\mathcal{S} = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_k$ of operator spaces is reflexive if and only if each of its direct summands is reflexive, but in general there is no relationship between reflexivity of $\mathcal{S}$ and its various subspaces. This is unfortunate since algebras generated by direct sums of operators are usually not full direct sums of algebras. Proposition 4.2 below provides a tool for dealing with this situation.

A vector $x \in V$ is called *separating* for the subspace $\mathcal{S} \subset L(V)$ if the map $s \to s\,x$ is injective on $\mathcal{S}$. It is easy to see that the existence of separating vectors survives the taking of direct sums. It follows that each singly generated algebra has a separating vector. In particular, existence of separating vectors does not guarantee reflexivity. We do, however, have the following basic result; see [2, Propositions 2.9, 3.2] for a proof.

Proposition 4.1. *Suppose a subspace $\mathcal{S}$ of $L(V)$ is reflexive and has a separating vector. Then every subspace of $\mathcal{S}$ is reflexive.*

The following proposition will be used to establish sufficiency in Theorem A. The proof was motivated by work of L. Ding [5],[6]. An operator $a = a_1 \oplus \cdots \oplus a_k$ in $L(V_1) \oplus \cdots \oplus L(V_k)$ is said to be *supported* on V_i if $a_j = 0$ for all $j \neq i$.

Proposition 4.2. *For each $i = 1, \ldots, k$, let $\mathcal{S}_i \subset L(V_i)$ be an operator space with separating vector x_i. Suppose $\mathcal{T}$ is a subspace of $\mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_k$, and for each i, write $\mathcal{T}_i = \{a \in \mathcal{T} : a \text{ is supported on } V_i\}$. Then $\mathcal{T}$ is reflexive if and only if each $\mathcal{T}_i$ is reflexive.*

Proof. We assume $k \geq 2$ to avoid trivialities. Note first that $x = x_1 \oplus \cdots \oplus x_k$ is a separating vector for $\mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_k$. For necessity, observe that x must also separate $\mathcal{T}$ whence the reflexivity of each $\mathcal{T}_i$ follows from that of $\mathcal{T}$ by Proposition 4.1.

For sufficiency, suppose $c = c_1 \oplus \cdots \oplus c_k \in Ref\mathcal{T}$. Since $c\,x \in \mathcal{T}(x)$, by subtracting an appropriate member of $\mathcal{T}$ from c if necessary, we may as well assume that $c\,x = 0$. We will show that $c_1 = 0$ whence $c = 0$ by symmetry, completing the proof.

Write $\bar{c} = c_2 \oplus \cdots \oplus c_k$ and $\bar{x} = x_2 \oplus \cdots \oplus x_k$. Given $y \in V_1$, we must have $(c_1 \oplus \bar{c})(y \oplus \bar{x}) = (s_y \oplus t_y)(y \oplus \bar{x})$ for some operator $s_y \oplus t_y \in \mathcal{T}$. This is equivalent to the two conditions

$$0 = \bar{c}\,\bar{x} = t_y\,\bar{x} \qquad \text{and} \qquad c_1\,y = s_y\,y.$$

Since $\bar{x}$ is separating, we first see that $t_y = 0$, which means that $s_y \oplus 0 \in \mathcal{T}_1$. The arbitrariness of y thus yields $c_1 \oplus 0 \in Ref\mathcal{T}_1$. Since $\mathcal{T}_1$ is reflexive, we have $c_1 \oplus 0 \in \mathcal{T}_1$. Since $x_1 \oplus 0$ separates $\mathcal{T}_1$ and $c_1\,x_1 = 0$, we get $c_1 = 0$ as desired.

We conclude this section with a trivial instance of Proposition 4.2, and a lemma which will allow more substantial applications later.

Corollary 4.3. *If a_1, a_2 are nilpotents of the same order then $a_1 \oplus a_2$ is reflexive.*

Proof. Take $\mathcal{S}_i$ to be the algebra generated by a_i and $\mathcal{T}$ the algebra generated by $a_1 \oplus a_2$. Apply the Proposition, noting that $\mathcal{T}_1 = \mathcal{T}_2 = \{0\}$.

The following lemma [2, Proposition 5.4] will play important role in the paper.

Lemma 4.4. *Every one–dimensional operator space is reflexive.*

5. Sufficiency. The first two propositions in this section record well-known facts; the second motivates our definition of simple N-tuple and is a special case of Proposition 5.4 below.

Proposition 5.1. *Let $\mathcal{A} \subset L(V)$ be a commutative algebra having a cyclic vector x. Then*

(1) *x is also a separating vector for $\mathcal{A}$,*
(2) *for each $c \in \mathcal{A}$, the rank of c is equal to the dimension of the ideal $c\,\mathcal{A}$.*

Proof. To see (1), note that by commutativity, if the cyclic vector x belongs to the kernel of an operator in $\mathcal{A}$ the whole space V is contained in that kernel. For (2), observe that the map $a \to a\,x$ defines a vector space isomorphism between $\mathcal{A}$ and V; for each $c \in \mathcal{A}$ it maps the ideal $c\mathcal{A}$ to range(c).

Proposition 5.2. *Let $a \in L(V)$ be nilpotent. Then the following are equivalent.*

(1) *the trivial operators 0 and I are the only idempotents commuting with a,*
(2) *the Jordan form of a is a single block,*
(3) *$\mathcal{A}(a)$ has a cyclic vector.*

Suppose $\boldsymbol{a} = (a_1, \ldots, a_N)$ and $\boldsymbol{b} = (b_1, \ldots, b_N)$ are N-tuples of operators acting on vector spaces V, W, respectively. Then we say $\boldsymbol{a}$ is *similar* to $\boldsymbol{b}$ if there is an invertible operator $s \in L(V, W)$ satisfying $b_i = s a_i s^{-1}$ for $i = 1, \ldots, N$.

All operators in the remainder of this section act on finite–dimensional *Hilbert* spaces. Recall that an N-tuple $\boldsymbol{a} = (a_1, \ldots, a_N)$ of operators is doubly commuting if $a_i a_j = a_j a_i$ and $a_i a_j^* = a_j^* a_i$ for each $i \neq j$. The condition is equivalent to requiring that the von Neumann algebras generated by the individual operators commute with each other.

An N–tuple $\boldsymbol{a} = (a_1, \ldots, a_N)$ of doubly commuting nilpotents is called *simple* if there are no non–trivial idempotents commuting with all of them.

Example 5.3. Let $\tilde{a}_i$ act on Hilbert space V_i for $i = 1, \ldots, N$ and form the tensor product space $V = V_1 \otimes \cdots \otimes V_N$. Define $a_i = I \otimes \cdots \otimes \tilde{a}_i \otimes \cdots \otimes I$. Then $\boldsymbol{a} = (a_1, \ldots, a_N)$ is doubly commuting.

In order for $\boldsymbol{a} = (a_1, \ldots, a_N)$ to be simple, it is necessary that the von Neumann algebras generated by the $\tilde{a}_i$ be factors. The condition fails to be sufficient, even when $N = 1$, not only because commuting projections may fail to be central, but also because non–self–adjoint idempotents must be taken into account.

Proposition 5.4. *Let $\boldsymbol{a} = (a_1, \ldots, a_N)$ be a doubly commuting N–tuple of nilpotents acting on a Hilbert space V. Then the following are equivalent.*

(1) *$\boldsymbol{a} = (a_1, \ldots, a_N)$ is simple,*
(2) *$\boldsymbol{a} = (a_1, \ldots, a_N)$ takes the form of Example 5.3 with each $\tilde{a}_i$ being simple,*
(3) *$\mathcal{A}(a_1, \ldots, a_N)$ has a cyclic vector.*

Proof. (1) $\Longrightarrow$ (2) The precise meaning of (2) involves a unitary map between the underlying Hilbert spaces. We argue by induction. If $N = 1$, there is nothing to prove. If $N \geq 2$, then the von Neumann algebra $\mathcal{N}(a_1)$ generated by a_1 must be a type I factor. Thus there are Hilbert spaces V_1 and K, and a unitary map $U : V \to V_1 \otimes K$ such that $U\mathcal{N}(a_1)U^{-1} = L(V_1) \otimes \mathbb{C}\, I_K$. There is no harm in suppressing U and assuming $\mathcal{N}(a_1) = L(V_1) \otimes \mathbb{C}\, I_K$, whence $a_1 = \tilde{a}_1 \otimes I_K$. By double commutativity, the von Neumann algebra $\mathcal{N}(a_2, \cdots, a_N)$ generated by $a_2, \cdots, a_N$ is contained in $I_{V_1} \otimes L(K)$. In particular, $\tilde{a}_1$ must be simple since $q \otimes I_K$ will commute with each a_i whenever q commutes with $\tilde{a}_1$. The decomposition is completed by applying the inductive hypothesis to $\mathcal{N}(a_2, \cdots, a_N)$.

(2) $\Longrightarrow$ (3) For each $i = 1, \ldots, N$, choose a cyclic vector x_i for $\mathcal{A}(\tilde{a}_i)$ and take $x = x_1 \otimes \cdots \otimes x_N$.

(3) $\Longrightarrow$ (1) Suppose q is an idempotent commuting with $a_1, \ldots, a_N$ and x is a cyclic vector for $\mathcal{A}(\boldsymbol{a})$. Then there is $c \in \mathcal{A}(\boldsymbol{a})$ with $q\,x = c\,x$. But then $\ker(q - c)$ contains all of $\mathcal{A}(\boldsymbol{a})\,x$ so $q = c$ belongs to $\mathcal{A}(\boldsymbol{a})$. Since 0 is the only operator which is simultaneously idempotent and nilpotent, we conclude that $q = 0$ or I, as desired.

Proposition 5.5. *Suppose $\boldsymbol{a} = (a_1, \ldots, a_N)$ is a simple N–tuple of doubly commuting nilpotents and $c \in \mathcal{A}(\boldsymbol{a})$.*

1. *All rank one members of $\mathcal{A}(\boldsymbol{a})$ are scalar multiples of one other.*
2. *If rank$(c) = 2$, then $c\mathcal{A}(\boldsymbol{a})$ is two-dimensional.*
3. *If rank$(c) \geq 2$, then $c\mathcal{A}(\boldsymbol{a})$ contains a member of rank two.*

Proof. To establish (1), write n_i for the order of a_i, and set $\boldsymbol{a}^{\boldsymbol{k}} = a_1^{k_1} \cdots a_N^{k_N}$ for each N–tuple $\boldsymbol{k} = (k_1, \ldots, k_N)$ of natural numbers. Suppose $c = \sum \lambda_{\boldsymbol{k}} \boldsymbol{a}^{\boldsymbol{k}}$ has rank one. Lemma 3.1 tells us that $a_i c = 0$ for each i. But, in view of Proposition 5.4 (2), we know that the operators $\boldsymbol{a}^{\boldsymbol{k}}$ for all $\boldsymbol{k} = (k_1, \ldots, k_N)$ with $0 \leq k_i \leq n_i - 1$ are linearly independent. Thus, $\lambda_{\boldsymbol{k}} = 0$ whenever $k_i \leq n_i - 2$. This forces c to be a scalar multiple of $a_1^{n_1-1} \cdots a_N^{n_N-1}$.

Part (2) is a consequence of Propositions 5.4 and 5.1(2).

We prove (3) inductively. Given rank$(c) > 2$, Proposition 5.1(2) yields dim $c\mathcal{A}(\boldsymbol{a}) > 2$ as well. On the other hand, $\mathcal{A}(\boldsymbol{a})$ is spanned by its nilpotent members and I. Thus there are nilpotent members b, d of $\mathcal{A}(\boldsymbol{a})$ such that cb, cd are independent. By Part (1) of the present proposition, at least one of these, say cd, has rank greater than one. Proposition 3.1 thus makes it possible to apply the inductive hypothesis to cd.

In general, similarities destroy double commutativity. Note however that tensor products of similarities on the underlying spaces V_i preserve double commutativity of the operators a_i of Example 5.3. This will be important in the following proof.

Proposition 5.6. *Every N–tuple $\boldsymbol{a} = (a_1, \ldots, a_N)$ of doubly commuting nilpotents is similar to an orthogonal direct sum of simple N–tuples.*

More precisely, there is a doubly indexed family $\{a_{ij} : i = 1, \ldots, N;\ j = 1, \ldots, k\}$ of nilpotents such that

1. *$(a_{1j}, \ldots, a_{Nj})$ is a simple N-tuple for each fixed j, and*
2. *the original operators a_i are simultaneously similar to the orthogonal direct sums $\oplus_{j=1}^{k} a_{ij}$.*

Proof. The von Neumann algebras $\mathcal{N}(a_1), \ldots, \mathcal{N}(a_N)$ commute so any self–adjont projection in the center of one of them will automatically commute with all of them. Doing a preliminary orthogonal decomposition we may thus assume all the $\mathcal{N}(a_i)$ to be factors. But then the proof of Proposition 5.4 ((1) $\Longrightarrow$ (2)) allows us to write $V = V_1 \otimes \cdots \otimes V_N$ and $a_i = I \otimes \cdots \otimes \tilde{a}_i \otimes \cdots \otimes I$ except that the $\tilde{a}_i$ need not be simple. Now the Jordan Canonical Form Theorem tells us that each $\tilde{a}_i$ is similar to an orthogonal direct sum of simple operators. Putting these similarities together, we can assume that the $\tilde{a}_i$ are themselves orthogonal direct sums of simple operators. The proof is completed by "splitting" these direct sums.

The sufficiency part of Theorem A can be stated as follows.

Theorem 5.7. *Suppose $\boldsymbol{a} = (a_1, \ldots, a_N)$ is an N–tuple of doubly commuting nilpotents acting on a finite–dimensional space Hilbert space V. If every rank two member of $\mathcal{A}(\boldsymbol{a})$ generates a one–dimensional ideal, then $\mathcal{A}(\boldsymbol{a})$ is reflexive .*

Proof. Apply Proposition 5.6 to write $a_i = \oplus_{j=1}^{k} a_{ij}$ where for each j, the N–tuple $(a_{1j}, \ldots, a_{Nj})$ acting on V_j is simple. For each j, take $\mathcal{S}_j$ to be $\mathcal{A}(a_{1j}, \cdots, a_{Nj})$ and set $\mathcal{T}_j = \{c \in \mathcal{A}(\boldsymbol{a}) : c \text{ is supported on } V_j\}$. Concentrating on $j = 1$ to simplify the notation, observe that every member of $\mathcal{T}_1$ takes the form $c_1 \oplus 0$ with c_1 belonging to the algebra $\mathcal{S}_1$. Applying Proposition 5.5 (2) and the hypothesis, we conclude that c_1 cannot have rank two. By Part (3) of the same Proposition, the rank of c_1 cannot exceed one. Finally Part (1) of the same Proposition implies that all such c_1 are scalar multiples of one another. By symmetry, each $\mathcal{T}_j$ is one-dimensional, and hence reflexive by Lemma 4.4. We complete the proof by applying Proposition 4.2 to $\mathcal{T} = \mathcal{A}(a_1, \ldots, a_N)$.

6. Jordan forms. In the present section, we will prove Theorem B and discuss some of its applications. In order to simplify the notation, we restrict attention to pairs of operators, but generalization to arbitrary N-tuples is routine. The *order* of a pair (a, b) of nilpotents is the pair of integers $(\text{order}(a), \text{order}(b))$.

We refer to the sequence of block sizes of a Jordan Canonical Form of an operator as a *Jordan sequence* for the operator; up to permutation, Jordan sequences provide a complete similarity invariant for single operators. Proposition 5.6 allows us to extend this notion to doubly commuting operator pairs.

Indeed, given a doubly commuting pair (a, b), apply Proposition 5.6 to obtain direct sums $\oplus a_{i=1}^{k}$, $\oplus b_{i=1}^{k}$ which are simultaneously similar to a, b such that for each i, the doubly commuting pair (a_i, b_i) is simple and acts on a Hilbert space V_i. Write (m_i, n_i) for the order of the simple pair (a_i, b_i). The finite sequence $(m_1, n_1), \ldots, (m_k, n_k)$ is referred to as a *Jordan sequence* of (a, b). Up to permutation, these sequences provide a complete similarity invariant for doubly commuting pairs.

Lemma 6.1. *Let (a, b) be a simple pair with order (m, n) and suppose $c \in \mathcal{A}(a, b)$. Then*

1. *rank$(c) \leq 1$ if and only if c is a scalar multiple of $a^{m-1}b^{n-1}$,*
2. *rank$(c) \leq 2$ if and only if c is a linear combination of $a^{m-2}b^{n-1}$, $a^{m-1}b^{n-2}$ and $a^{m-1}b^{n-1}$.*

Proof. (1) is a consequence of Proposition 5.5 and 5.1 (2).

For sufficiency of (2), observe that if c takes this form, then ac and bc are both scalar multiples of $a^{m-1}b^{n-1}$, so $\dim(c\mathcal{A}(a,b)) \leq 2$. Thus $\mathrm{rank}(c) \leq 2$ by Proposition 5.1(2).

For the converse, suppose $c = \sum_{i,j=0}^{m-1,n-1} \lambda_{ij} a^i b^j$ has rank two. By Lemma 3.1, we have $\mathrm{rank}(a\,c) < \mathrm{rank}(c) = 2$. Applying part (1), $a\,c = \sum_{i,j=0}^{m-1,n-1} \lambda_{ij} a^{i+1} b^j = \alpha a^{m-1} b^{n-1}$ for some $\alpha \in \mathbb{C}$. Thus $\lambda_{ij} = 0$ for $i < m-2$ and $\lambda_{m-2,j} = 0$ for $j < n-1$. By symmetry, we have the desired form.

Lemma 6.1 admits a partial generalization in that every $c \in \mathcal{A}(a,b)$ of rank r or less must be a linear combination of $\{a^{m-i}b^{n-j} : 0 < i, 0 < j, i+j \leq r+1\}$. To see that the condition is not sufficient, note that if (a,b) is a simple pair of order (2,2) then $\mathcal{A}(a,b)$ does not contain any members of rank three.

When $b = 0$, the first three conditions of the following theorem reduce to the corresponding conditions of Section 2. The equivalence of (1) and (4) is Theorem B of the Introduction.

Theorem 6.2. *Suppose (a,b) is a double commuting pair with corresponding Jordan sequence $(m_1,n_1),\ldots,(m_k,n_k)$. Then the following are equivalent.*

(1) *For each index i,*
   if $m_i \geq 2$, we can find $j \neq i$ with $m_j \geq m_i - 1$ and $n_j \geq n_i$, and
   if $n_i \geq 2$, we can find $j \neq i$ with $n_j \geq n_i - 1$ and $m_j \geq m_i$.

(2) *If $c = \oplus_{i=1}^k c_i \in \mathcal{A}(a,b)$ has rank 2, then $c_i \neq 0$ for two values of i.*

(3) *If $c \in \mathcal{A}(a,b)$ has rank 2, then $c\mathcal{A}(a,b)$ is one–dimensional.*

(4) *$\mathcal{A}(a,b)$ is reflexive.*

Proof. (1) $\Longrightarrow$ (2) Arguing contrapositively, assume for definiteness that $c = c_1 \oplus 0$ in $\mathcal{A}(a,b)$ has rank two and is supported on V_1. Write $c = \sum \lambda_{hl} a^h b^l$. On the one hand, $\sum \lambda_{hl} a_1^h b_1^l$ has rank two, so in view of Lemma 6.1(2), we may as well assume that $\lambda_{m_1-2,n_1-1} \neq 0$. On the other hand, for $j \neq 1$, the vanishing of c_j forces $m_j \leq m_1 - 2$ or $n_j \leq n_1 - 1$. This means that (1) fails for $i = 1$ and completes the proof.

(2) $\Longrightarrow$ (3) Suppose $c \in \mathcal{A}(a,b)$ has rank 2. By (2) may assume $c = c_1 \oplus c_2 \oplus 0$ with c_1, c_2 of rank 1. By Lemma 3.1, we have $a_1\,c_1 = a_2\,c_2 = 0$ so $a\,c = 0$. Similarly $b\,c = 0$. Thus $c\mathcal{A}(a,b)$ is one–dimensional.

(3) $\Longleftrightarrow$ (4) This is Theorems 5.7 and 3.3.

(3) $\Longrightarrow$ (1) Suppose $m_1 \geq 2$. By Lemma 6.1, the rank of the operator $a_1^{m_1-2}b_1^{n_1-1}$ is precisely two and hence by Proposition 5.1(2), it generates a two-dimensional ideal. The assumption (3) rules out the possibility that $a^{m_1-2}b^{n_1-1}$ be supported on V_1. In other words, $m_j > m_1 - 2$ and $n_j > n_1 - 1$ for some $j \neq 1$. This establishes the first half of (1) when $i = 1$ and the rest follows by symmetry.

It is convenient to call the pair (m_i, n_i) *majorized* if Condition (1) of Theorem 6.2 is fulfilled for the index i. The discussion of examples is also facilitated by calling a Jordan sequence *reflexive* if the corresponding operator algebra is reflexive. As a first application of Theorem 6.2, we review the examples given in the introduction.

Example 6.3. The algebra denoted by (*) is generated by a pair with Jordan sequence (2,2), (3,1), (1,2), while a generating pair for the algebra (**) has Jordan sequence (2,2), (3,1), (2,1). It is easy to check that each term in the first sequence is majorized, but the term (2,2) has no majorant in the second sequence.

The next two results apply Theorems A and B to tensor products.

Proposition 6.4. *Suppose a, b are nilpotent. Then $\mathcal{A}(a \otimes b)$ is reflexive.*

Proof. Suppose $c = p(a \otimes b) \in \mathcal{A}(a \otimes b)$ has rank two. Factor the polynomial p to obtain $p(X) = X^k q(X)$, with $q(0) \neq 0$. Then $q(a \otimes b)$ is invertible so in fact $(a \otimes b)^k$ has rank two. But then $(\text{rank}(a^k))(\text{rank}(b^k)) = 2$. In particular, either a^k or b^k has rank one. In either case, $c(a \otimes b) = (a^{k+1} \otimes b^{k+1}) q(a \otimes b) = 0$. Therefore, c generates a one–dimensional ideal and Theorem A applies.

The operators c, d appearing in the next result are not assumed to be simple.

Corollary 6.5. *Suppose c, d are nilpotent operators. If c and d are reflexive, then the algebra $\mathcal{A}(c \otimes I, I \otimes d)$ is reflexive. If the two largest members of the Jordan sequence of c or of d are the same, then the algebra $\mathcal{A}(c \otimes I, I \otimes d)$ is reflexive. In all other cases, the algebra $\mathcal{A}(c \otimes I, I \otimes d)$ is not reflexive.*

Proof. The Jordan sequence of $(c \otimes I, I \otimes d)$ is the cartesian product of the Jordan sequences of c and d. Let $m_1 \geq m_2$ and $n_1 \geq n_2$ denote the two largest terms in the Jordan sequences of c and d, respectively. All elements of the Jordan sequence of $(c \otimes I, I \otimes d)$ except (m_1, n_1) are majorized. The remaining term (m_1, n_1) is majorized in precisely the following situations:

(1) $m_1 = m_2$ or
(2) $n_1 = n_2$ or
(3) $m_1 = m_2 + 1$ and $n_1 = n_2 + 1$.

These correspond to the cases listed in the statement of the Corollary.

Example 6.6. In the single operator case, there are always at least three singleton sequences whose concatenations with a given Jordan sequence produce reflexive sequences. For example, the singleton sequence 5 can be lengthened to the reflexive sequences 5, 4 ; 5, 5 ; and 5, 6. On the other hand, the only two–term reflexive extension of the Jordan sequence (5,7) is (5,7), (5,7). Even the three–term reflexive extensions of (5,7) are limited. For example, only the first of the following four extensions of (5,7) is reflexive:

$$(4,7),(5,7),(5,6); \quad (4,6),(5,7),(5,6); \quad (5,8),(5,7),(5,6); \quad (6,6),(5,7),(8,4).$$

Example 6.7. The Jordan sequence $(1,10),(2,9),(3,8),(4,7),(4,6)$ represents a reflexive pair and it is minimal in the sense that none of its proper subsequences is reflexive. This contrasts with the single operator case where discarding all but the two largest terms of a Jordan sequence does not affect reflexivity.

7. Generalizations and concluding remarks.

The first topic of this section is a Hilbert space free version of Theorems A and B.

The reader is referred to [1] or [9] for background in ring theory, in particular for the Wedderburn Structure Theory used below. We recall the relevant definitions here. A left module over a ring is said to be *simple* if it has no non-trivial submodules; it is *semisimple* if it can be expresed as a direct sum of simple modules. A ring is *simple* if it has no non–trivial two–sided ideals; it is *semisimple* if it is semisimple when regarded as a left module over itself.

We need the following well–known fact.

Proposition 7.1. *An operator algebra is semisimple if and only if it is similar to a von Neumann algebra.*

Proof. Let $\mathcal{B}$ be a subalgebra of $L(V)$. From the ring–theoretic point of view, the underlying vector space V is a (faithful, left) module over $\mathcal{B}$. It is clear from the definitions that simplicity and semisimplicity are invariant under similarity. Since we are restricting attention to finite–dimensional vector spaces, $\mathcal{B}$ is semisimple if and only if it is a direct sum of simple operator algebras.

It is easy to check that the full algebra $L(V)$ is simple. The proof of sufficiency is completed by appealing to the known structure of von Neumann algebras as direct sums of factors.

For the converse, recall that the only finite–dimensional division algebra over the complex numbers is $\mathbb{C}$ itself. Suppose first that $\mathcal{B}$ is a simple operator algebra. The Wedderburn Structure Theorems then tell us that $\mathcal{B}$ is ring isomorphic to some full operator algebra $L(W)$, in fact that $\mathcal{B}$ is spatially isomorphic (i.e. similar) to $L(W) \otimes \mathbb{C}I_K$ for some auxiliary vector space K. The last algebra can be made into a von Neumann algebra by introducing an appropriate inner product on the underlying space $W \otimes K$. To complete the proof for semisimple $\mathcal{B}$, apply the preceding construction to its direct summands, taking care to define the inner product to make its corresponding direct summands of the underlying space mutually orthogonal.

An N–tuple $\boldsymbol{a} = (a_1, \ldots, a_N)$ of operators acting on a common vector space is called *semisimple* if the a_i belong to mutually commuting semisimple algebras.

A semisimple N–tuple $\boldsymbol{a} = (a_1, \ldots, a_N)$ of nilpotents is called *simple* if only the trivial idempotents commute with all of them.

The following are immediate consequences of Proposition 7.1.

Corollary 7.2. *An N–tuple $\boldsymbol{a} = (a_1, \ldots, a_N)$ of nilpotents is semisimple if and only if it is similar to a doubly commuting N–tuple.*

Proposition 7.3. *Theorems A , B, 5.7, and 6.2 remain valid when the assumption of double commutativity is replaced by semisimplicity.*

Our final generalization of Theorem A removes the hypothesis of nilpotence.

Theorem 7.4. *In order for a commutative operator algebra $\mathcal{A}$ to be reflexive, it is necessary that for each rank two member c, there is an idempotent $q \in \mathcal{A}$ such that qc generates a one–dimensional ideal. If the underlying vector space is finite–dimensional and $\mathcal{A}$ has a set of generators belonging to mutually commuting semisimple algebras, then this condition is also sufficient.*

Proof. All properties mentioned in the Theorem hold for a full direct sum of operator algebras if and only if they hold for each direct summand. Thus we may as well assume that $\mathcal{A}$ contains only the trivial idempotents. In the latter situation, however, the Theorem reduces to the Hilbert space free version of Theorem A.

Remark 7.5. In Proposition 6.4, we showed that the tensor product of two nilpotent operators is always reflexive. As discussed in [7], the situation is much more complicated when the hypothesis of nilpotence is dropped. From the point of view of Theorem 7.4, the problem arises from the failure of idempotents in $\mathcal{A}(a \otimes b)$ corresponding to points in $\sigma(a \otimes b)$ to be simple tensor products of idempotents in $\mathcal{A}(a)$ and $\mathcal{A}(b)$.

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