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SPECTRAL MULTIPLICITY FOR TENSOR PRODUCTS OF NORMAL OPERATORS

EDWARD A. AZOFF¹

ABSTRACT. Two normal operators N_1 and N_2 are constructed such that for any pair m_1 and m_2 of their respective multiplicity functions, the 'convolution' $(m_1 * m_2)(\lambda) \equiv \sum \{m_1(\lambda_1) \cdot m_2(\lambda_2) | \lambda_1 \cdot \lambda_2 = \lambda\}$ fails to be a multiplicity function for the tensor product $N_1 \otimes N_2$.

Introduction. All operators discussed in this paper will be normal operators acting on separable Hilbert spaces. Let N be such an operator and let ν be one of its scalar spectral measures. The theory of spectral multiplicity associates with N a function $m: \mathbb{C} \rightarrow \mathbb{N} \cup \{\infty\}$ such that the equivalence class $[m]_\nu$ (i.e. the family of functions: $\mathbb{C} \rightarrow \mathbb{N} \cup \{\infty\}$ agreeing with m off a ν -null set) provides a complete unitary invariant for N . In particular, the spectrum of N and the measure class of ν are completely determined by $[m]_\nu$. On the other hand, it is only the equivalence class $[m]_\nu$, and not m , that is uniquely determined by N . It would be nice to choose 'canonical' representatives for these equivalence classes. For operators with countable spectra this is possible: we simply take m to be the unique representative of its equivalence class which is supported on the point spectrum of N .

If N_i ($i = 1, 2$) have countable spectra and the canonical multiplicity functions m_i ($i = 1, 2$) are chosen as in the preceding paragraph, then the function $m_1 * m_2$ defined by

$$(m_1 * m_2)(\lambda) \equiv \sum \{m_1(\lambda_1) \cdot m_2(\lambda_2) | \lambda_1 \cdot \lambda_2 = \lambda\} \quad (1)$$

provides a multiplicity function for the tensor product $N_1 \otimes N_2$. In their paper [1], M. B. Abrahamse and T. L. Kriete constructed 'canonical' multiplicity functions for multiplication operators. They then asked whether formula (1) continues to provide a multiplicity function for the tensor product of multiplication operators with 'canonical' multiplicity functions m_1 and m_2 respectively. Example 2 of this note settles this question negatively. One might think that this could be rectified by changing the Abrahamse-Kriete multiplicity function. Example 3 of this note exhibits two multiplication operators N_1 and N_2 such that for every choice of multiplicity functions, equation (1) fails to provide a multiplicity function for $N_1 \otimes N_2$. This can be interpreted as meaning that there is, in general, no canonical way to choose representatives for multiplicity classes of normal operators.

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Notation and preliminaries. We recall the basic components of multiplicity theory for normal operators. For a fuller discussion, the reader should consult the introductory sections of [1] or [2].

Let N be a normal operator with spectral measure E . A measure ν on \mathbf{C} is said to be a *scalar spectral measure* for N in case $E(S) = 0$ if and only if $\nu(S) = 0$ for S a Borel set in \mathbf{C} . (All measures discussed in this note are completions of probability measures defined on complete separable metric spaces.) There exists a direct integral $\int_{\mathbf{C}}^{\oplus} H(\lambda) d\nu$ of Hilbert spaces having ν as a scalar spectral measure for N such that N is unitarily equivalent to the operator M on $\int_{\mathbf{C}}^{\oplus} H(\lambda) d\nu$ defined by $Mf(\lambda) = \lambda f(\lambda)$. The function $m: \mathbf{C} \rightarrow \mathbf{N} \cup \{\infty\}$ defined by $m(\lambda) = \dim H(\lambda)$ is called a *multiplicity function* for N ; its equivalence class $[m]_{\nu}$ is called the *multiplicity class* of N and constitutes a complete unitary invariant for N .

Let (X, μ) be a measure space. As usual, we write $L^{\infty}(\mu)$ ($L^2(\mu)$) for the collection of equivalence classes of bounded measurable (respectively square-integrable) complex-valued functions on X . Let $\phi \in L^{\infty}(\mu)$. Then the multiplication operator M_{ϕ} is defined by $M_{\phi}f = \phi f$ ($f \in L^2(\mu)$). It is a bounded normal operator having $\nu \equiv \mu \circ \phi^{-1}$ as a scalar spectral measure. The papers [1] and [2] address the problem of constructing multiplicity functions for such operators. We close this section by summarizing these results.

If φ is a Borel representative of ϕ , we define $m_{\varphi}: \mathbf{C} \rightarrow \mathbf{N} \cup \{\infty\}$ by taking $m_{\varphi}(\lambda)$ to be the cardinality of the set $\varphi^{-1}(\lambda)$. (All infinite cardinalities are identified.)

PROPOSITION 1. *Let $\phi \in L^{\infty}(X, \mu)$, take M_{ϕ} to be the corresponding multiplication operator, and set $\nu = \mu \circ \phi^{-1}$.*

(1) *There is a Borel representative φ_0 of ϕ such that m_{φ_0} is a multiplicity function for M_{ϕ} .*

(2) *If φ is any Borel representative of ϕ , then m_{φ} is ν -measurable and $m_{\varphi}(\lambda) \geq m_{\varphi_0}(\lambda)$ for ν -almost all $\lambda \in \mathbf{C}$.*

(3) *Given a Borel multiplicity function m for M_{ϕ} , there is a Borel representative φ of ϕ such that $m_{\varphi}(\lambda) \leq m(\lambda)$ for all $\lambda \neq 0$.*

(4) *In order that M_{ϕ} have uniform multiplicity one, it is necessary and sufficient that some (hence every) representative φ of ϕ be one-to-one on a set of full μ -measure.*

PROOF. Statements (1) and (2) constitute Theorem 4.1 of [2]; statement (4) is contained in Corollary 4.1 of that paper. To prove (3), choose φ_0 as in (1), and set $E = \{\lambda \in \mathbf{C} | m_{\varphi_0}(\lambda) > m(\lambda)\}$. Since $\nu(E) = 0$, there is a Borel set F containing E with $\nu(F) = 0$. Set

$$\varphi(x) = \begin{cases} \varphi_0(x) & \text{if } x \notin \varphi_0^{-1}(F), \\ 0 & \text{if } x \in \varphi_0^{-1}(F). \end{cases}$$

For $\lambda \neq 0$, we have $\varphi^{-1}(\lambda) \subseteq \varphi_0^{-1}(\lambda)$ so $m_{\varphi}(\lambda) \leq m_{\varphi_0}(\lambda)$. Moreover, if $\lambda \in E \sim \{0\}$, $\varphi^{-1}(\lambda) = \emptyset$ so $m_{\varphi}(\lambda) = 0$. This completes the proof. \square

REMARK. If ν is not totally atomic the Borel representative φ of (3) can be chosen to satisfy $m_{\varphi} \equiv m$.

Finally we recall the construction of the Abrahamse-Kriete multiplicity function [1] for M_ϕ . Given $\lambda \in \text{ess ran } \phi$, a point $x \in X$ is said to belong to the *essential preimage*, $\phi_{\text{ess}}^{-1}(\lambda)$, of ϕ at λ in case

$$\lim_{\delta \rightarrow 0} \frac{\mu(V \cap \phi^{-1}(B_\delta(\lambda)))}{\mu(\phi^{-1}(B_\delta(\lambda)))}$$

is strictly positive for every neighborhood V of x ; here $B_\delta(\lambda)$ denotes the closed ball of radius δ about λ . The function $\bar{m}_\phi: \mathbb{C} \rightarrow \mathbb{N} \cup \{\infty\}$ (or simply \bar{m} if ϕ is understood) is defined as

$$\bar{m}(\lambda) = \begin{cases} \text{cardinality of } \phi_{\text{ess}}^{-1}(\lambda) & \text{if } \lambda \in \text{ess ran } \phi, \\ 0 & \text{if } \lambda \notin \text{ess ran } \phi. \end{cases}$$

It is shown in [1] that \bar{m} is a multiplicity function for M_ϕ ; in the sequel, it will be referred to as the *Abrahamse-Kriete multiplicity function* for M_ϕ .

Noncanonicity of multiplicity functions. Given functions $m_i: \mathbb{C} \rightarrow \mathbb{N} \cup \{\infty\}$ ($i = 1, 2$), we adopt equation (1) as the definition of $m_1 * m_2$.

PROPOSITION 2. *Let N_1 and N_2 be normal operators having multiplicity functions m_1 and m_2 respectively. Let ν be a scalar spectral measure for $N_1 \otimes N_2$, and suppose m is one of its multiplicity functions. Then $(m_1 * m_2)(\lambda) \geq m(\lambda)$ for ν -almost all λ .*

PROOF. Let ν_i be a scalar spectral measure for N_i ; up to absolute continuity, the measure ν is given by $\nu(E) = (\nu_1 \times \nu_2)\{(\lambda_1, \lambda_2) \in \mathbb{C} \times \mathbb{C} | \lambda_1 \cdot \lambda_2 \in E\}$. By lowering the values of m_i on a ν_i -null set, we can make m_1 and m_2 Borel measurable. Since this can only decrease the value of $(m_1 * m_2)(\lambda)$, we may as well assume m_1 and m_2 are Borel to start with. Since every normal operator is unitarily equivalent to a multiplication operator, we may take N_i to be a multiplication operator M_{ϕ_i} acting on a Hilbert space $L^2(X_i, \mu_i)$. Apply Proposition 1(3) to construct a Borel representative φ_i of ϕ_i such that $m_{\varphi_i}(\lambda) \leq m_i(\lambda)$ for $\lambda \neq 0$, let $\varphi: X_1 \times X_2 \rightarrow \mathbb{C}$ by $\varphi(x_1, x_2) = \varphi_1(x_1) \cdot \varphi_2(x_2)$ and write $\phi = [\varphi]_{\mu_1 \times \mu_2}$. Then $N_1 \otimes N_2$ is unitarily equivalent to M_ϕ acting on $L^2(X_1 \times X_2, \mu_1 \times \mu_2)$ and so $m(\lambda) \leq m_\varphi(\lambda)$ for ν -almost all λ by Proposition 1(2). But for $\lambda \neq 0$, we have $m_\varphi(\lambda) = (m_{\varphi_1} * m_{\varphi_2})(\lambda) \leq (m_1 * m_2)(\lambda)$ so the proof is complete as long as $\{0\}$ is not an atom for ν . On the other hand, if $\{0\}$ is an atom for ν , then $m(0) = \dim \ker(N_1 \otimes N_2)$ which is easily seen to be less than or equal to $m_1 * m_2(0)$. \square

Proposition 2 means that $m_1 * m_2$ can only fail to be a multiplicity function for $N_1 \otimes N_2$ by being too large, i.e. by having the strict inequality $(m_1 * m_2)(\lambda) > m(\lambda)$ hold on a set of λ which does not have ν measure zero. The three examples which follow illustrate successively more pathological instances of this behavior.

In each of these examples, μ_i ($i = 1, 2$) is a measure supported on $[0, 1]$, $\varphi_1 = \varphi_2$ is the identity function on $[0, 1]$, $\phi_i = [\varphi_i]_{\mu_i}$, and N_i is the multiplication operator M_{ϕ_i} acting on $L^2([0, 1], \mu_i)$. Thus $N_1 \otimes N_2$ is unitarily equivalent to the multiplication operator $M_{\phi_1 \otimes \phi_2}$ acting on $L^2([0, 1] \times [0, 1], \mu_1 \times \mu_2)$. It will be convenient to let $p: \mathbb{R}^2 \rightarrow \mathbb{R}$ denote multiplication. Of course, μ_i is a scalar spectral measure for N_i and $\nu \equiv (\mu_1 \times \mu_2) \circ p^{-1}$ provides a scalar spectral measure for $N_1 \otimes N_2$.

EXAMPLE 1. Let $\mu_1 = \mu_2$ be the point-mass measure based at 1, and let $m_1 = m_2: \mathbb{C} \rightarrow \mathbb{N}$ be the constant function 1. Then m_i is a multiplicity function for N_i , but $m_1 * m_2 \equiv \infty$ is not a multiplicity function for $N_1 \otimes N_2 = I$. On the other hand, $\bar{m}_1 = \bar{m}_2$ equals the characteristic function of the singleton set $\{1\}$, so $\bar{m}_1 * \bar{m}_2$ is a multiplicity function for $N_1 \otimes N_2$.

EXAMPLE 2. Let $A_1 = \{e^{-q} | q > 0, q \in \mathbb{Q}\}$ and $A_2 = \{e^{-q\sqrt{2}} | q > 0, q \in \mathbb{Q}\}$. Let μ_i be a totally atomic measure on $[0, 1]$ having A_i as its set of atoms. Since A_i is dense in $[0, 1]$, the Abrahamse-Kriete multiplicity function \bar{m}_i turns out to be the characteristic function $X_{[0,1]}$. Thus $(\bar{m}_1 * \bar{m}_2)(\lambda) = \infty$ for each $\lambda \in (0, 1)$. On the other hand, $\mu_1 \times \mu_2$ is supported on $A_1 \times A_2$ and p is one-to-one on $A_1 \times A_2$, so $N_1 \otimes N_2$ has uniform multiplicity one. Thus $\bar{m}_1 * \bar{m}_2$ is not a multiplicity function for $N_1 \otimes N_2$, and we have a counterexample to the conjecture in §7 of [1]. Note however that $m_i \equiv X_{A_i}$ provides a multiplicity for N_i such that $m_1 * m_2$ is a multiplicity function for $N_1 \otimes N_2$.

EXAMPLE 3. We construct measures μ_i ($i = 1, 2$) such that for any choice m_i of multiplicity function for N_i , $m_1 * m_2$ fails to be a multiplicity function for $N_1 \otimes N_2$. We need the following lemma.

LEMMA. *There exist four uncountable, closed subsets A, B, C, D of $[0, 1]$ with the following properties:*

- (1) p is one-to-one on $(A \times D) \cup (B \times C) \cup (B \times D)$.
- (2) For each $b \in B$, there is a unique $d \in D$ such that $b \cdot d \in A \cdot C$ and for each $d \in D$, there is a unique $b \in B$, such that $b \cdot d \in A \cdot C$.
- (3) $A \cdot C \subseteq B \cdot D$.

PROOF. For a member $x \in [0, 10]$, we let x_n ($n = 0, 1, 2, \dots$) denote the n th digit in its decimal expansion. (Expansions ending in nines will not be considered.) Let A_0, B_0, C_0, D_0 be the sets of numbers having the decimal expansions indicated in the following table.

	x_0	x_{2n-1} ($n > 0$)	x_{2n} ($n > 0$)
A_0	1	0 or 1	1
B_0	0	0	0, 1, 3 or 4
C_0	3	0 or 3	3
D_0	4	0, 1, 3 or 4	$4 - x_{2n-1}$

It is easy to check that A_0, B_0, C_0, D_0 satisfy conditions analogous to (1), (2) and (3) of the lemma with addition in place of multiplication. Set $A = \{e^{-x} | x \in A_0\}$, $B = e^{-B_0}$, $C = e^{-C_0}$, $D = e^{-D_0}$. \square

By Theorem 8.1 of [3], we know that the sets A and C can be equipped with nonatomic measures α and γ respectively. Define a measure β on B by setting $\beta(S) = (\alpha \times \gamma)(p^{-1}(S \cdot D))$. Similarly let δ be defined on D by $\delta(S) = (\alpha \times \gamma)(p^{-1}(B \cdot S))$. Finally we take $\mu_1 = (\alpha + \beta)/2$, $\mu_2 = (\gamma + \delta)/2$.

We first show that $N_1 \otimes N_2$ has uniform multiplicity 1. To see this, let $E = B \times D \cap p^{-1}(A \cdot C)$. By (2) of the lemma, each section $E_d \equiv \{b \in B | (b, d) \in E\}$ of E in a singleton, and hence has μ_1 measure zero. It follows by Fubini's theorem

that $\mu_1 \times \mu_2(E) = 0$. By (1) and (3) of the lemma, p is one-to-one on $(A \cup B) \times (C \cup D) \sim E$. Since $\mu_1 \times \mu_2$ is supported on this set, we conclude by Proposition 1(4) that $N_1 \otimes N_2$ has uniform multiplicity one.

Suppose next that m_1 and m_2 are multiplicity functions for N_1 and N_2 respectively. Since μ_i is a scalar spectral measure for N_i , there is a set X_i of full measure in $[0, 1]$ on which m_i is identically one. If λ belongs to both of the sets $(X_1 \cap A) \cdot (X_2 \cap C)$ and $(X_1 \cap B) \cdot (X_2 \cap D) \cap A \cdot C$, then $(m_1 * m_2)(\lambda) \geq 2$. We will show that these sets have full ν -measure in $A \cdot C$. Now

$$\nu(A \cdot C) = (\mu_1 \times \mu_2)(p^{-1}(A \cdot C)) = (\mu_1 \times \mu_2)(A \times C \cup E) = \frac{1}{4}.$$

Thus we will know that $(m_1 * m_2)(\lambda) > 1$ on a set of positive ν -measure, so that in view of the preceding paragraph, $m_1 * m_2$ cannot be a multiplicity function for $N_1 \otimes N_2$.

It remains to show that

$$\nu[(X_1 \cap A) \cdot (X_2 \cap C)] = \nu[(X_1 \cap B) \cdot (X_2 \cap D) \cap A \cdot C] = \frac{1}{4}.$$

The first set is easy: $\nu((X_1 \cap A) \cdot (X_2 \cap C)) \geq (\mu_1 \times \mu_2)[(X_1 \cap A) \times (X_2 \cap C)] = \mu_1(A) \cdot \mu_2(C) = \frac{1}{4}$. To handle the second set, we first note that if $S \subseteq A \times C$, then $(\mu_1 \times \mu_2)(S) = \frac{1}{4}(\alpha \times \gamma)(S)$. It follows that $\nu[(B \sim X_1) \cdot D \cap A \cdot C] = \frac{1}{4}(\alpha \times \gamma)(p^{-1}[(B \sim X_1) \cdot D])$ which is by definition of β equal to $\frac{1}{4}\beta(B \sim X_1)$. Since X_1 has full measure, we conclude that $\nu[(B \sim X_1) \cdot D \cap A \cdot C] = 0$. A similar argument shows that $\nu[B \cdot (D \sim X_2) \cap A \cdot C] = 0$. Thus

$$\nu[(X_1 \cap B) \cdot (X_2 \cap D) \cap A \cdot C] = \nu(B \cdot D \cap A \cdot C) = \nu(A \cdot C) = \frac{1}{4}$$

as desired.

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