A GOOD SIDE TO NON–REFLEXIVE TRANSFORMATIONS

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To Carl Pearcy in honor of his 60th birthday

Abstract. Let V be a vector space over an infinite scalar field and suppose that $a \in L(V)$. We show that, as a strictly closed algebra of transformations, $\operatorname{alg} \operatorname{lat} a$ is generated by its own rank one members and the original transformation a. To do so, we obtain a concrete description of the rank one members of $\operatorname{alg} \operatorname{lat} a$ when a is locally algebraic.

Applications include a unified approach to earlier reflexivity results and an explanation of the phenomenon that among locally nilpotent transformations, it is the *non–reflexive* ones which always admit reasonable Jordan canonical forms.

1. Introduction

Early on in our linear algebra experience, we are taught to regard invariant subspaces as "good"—operators with invariant subspaces admit block triangular representations and this is thought of as a first step in developing a structure theory for such operators. The Jordan canonical form theorem is a notable success of this point of view; basic blocks come from complemented invariant subspaces and the internal structure of these blocks reflects chains of further invariant subspaces.

An operator a is said to be reflexive if each operator leaving invariant all a–invariant subspaces of the underlying domain space must belong to the (suitably closed) operator algebra generated by a. From the perspective of the preceding paragraph, such operators should be particularly "simple". Indeed, many of the bounded Hilbert space operators we understand best— normal and toeplitz operators—are reflexive. More recently, in Theorem 10.6 of [3], H. Bercovici, C. Foias, and C. Pearcy have shown that every weighted shift whose norm coincides with its spectral radius must be reflexive.

We are, of course, a long way from any global structure theory for bounded Hilbert space operators—reflexive or otherwise. The situation is different in the purely algebraic setting when the underlying vector space V is not equipped with any topology. In fact, if $\dim V = \aleph_0$, a theorem of Ulm featured in Section 11 of I. Kaplansky's monograph [12] provides a complete set of similarity invariants for the locally algebraic operators on V. Since recent work of M.B. Delaï [6] characterizes the reflexive transformations on such spaces, we are in the position of deciding whether the reflexive transformations are indeed simpler than the non–reflexive ones. The answer turns out to be no in a surprisingly strong sense.

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Let V be a vector space over an infinite field. Given $a \in L(V)$, we write $\operatorname{alg}\operatorname{lat} a$ for the algebra of all transformations on V which leave invariant each a–invariant subspace of V. Following Delaï, we call $a \in L(V)$ *reflexive* if each member of $\operatorname{alg}\operatorname{lat} a$ must belong to the *strictly closed* algebra generated by a. A surprising result due to D. Hadwin [10] states that every transformation which is *not* locally algebraic must be reflexive. This is a *qualitative* blow against the simplicity of reflexive transformations.

The coup de grâce comes from Delaï's analysis of locally nilpotent transformations. Indeed, Theorem 6 of [6] shows that if such a transformation fails to be reflexive, some quotient space $\ker a^{n+1}/\ker a^n$ must be one–dimensional. In the last section of the present paper, we apply techniques from Kaplansky's monograph to show that this condition forces a to be a direct sum of (finite and infinite) backward shifts. In particular, the Ulm invariants of these transformations stabilize at the first infinite ordinal. By contrast, there are (necessarily reflexive) locally nilpotent transformations whose Ulm invariants take arbitrarily long to stabilize; this is *quantitative* evidence for the superiority of *non*–reflexive transformations.

The body of this paper attempts to explain this mystery. In operator theory, associating rank one operators with an object of study is often a good way to understand the structure of that object. To cite a few sample references, the analysis of nest–like algebras in [9], [13] is facilitated by the fact that they *contain* many rank one members, while the study of dual algebras in [3] makes extensive use of those members of their preduals which have rank one representatives. More to the point of the present paper, an algebra of bounded Hilbert space operators is reflexive if and only if rank one operators are weak*–total in its preannihilator.

Returning to the setting of a locally nilpotent transformation a, the algebra it generates contains few if any rank one members. $\operatorname{alg}\operatorname{lat} a$ can, however, be richer in rank one members. In fact, it is a corollary of our main result that this happens precisely when a is non–reflexive. Evidently, $\operatorname{alg}\operatorname{lat} a$ is more "closely related" to a than any putative "preannihilator", whence rank one members of the former yield more information about the structure of a than rank one members of the latter.

The structure of the balance of the paper is as follows. In Section 2, we review the basic definitions in the setting of linear subspaces of $L(V)$; this highlights the contrast between general and singly generated subalgebras of $L(V)$—a contrast which does not exist in the Hilbert space setting. Section 3 includes a self–contained exposition of earlier work from [5], [10], and [6], and a brief comparison with the situation for bounded Hilbert space operators.

In Sections 4 and 5, we concentrate on primary transformations: $a \in L(V)$ having the property that $p(a)$ is locally nilpotent for some irreducible polynomial p. A simple set of "block invariants" for such transformations is introduced and studied in Section 4. These are used to concretely characterize the rank one members of $\operatorname{alg}\operatorname{lat} a$ in Theorem 5.4. In Theorem 5.8 we obtain our main result that $\operatorname{alg}\operatorname{lat} a$ is always generated by its rank one members together with a. In particular, this approach explains the anomaly that the characterization of when a is reflexive is most complicated when p is linear and a is algebraic.

The final section of the paper fills in the details of our earlier discussion concerning the relative simplicity of non-reflexive local nilpotents as compared with their reflexive counterparts.

2. Reflexive closures

Throughout this paper, V will denote a vector space over an infinite field F; we write $L(V)$ for the algebra of all linear transformations on V. As usual, V^* denotes the dual space of V. Given $v \in V$ and $\phi \in V^*$, we write $v \otimes \phi$ for the transformation defined by

$$(v \otimes \phi)(y) = \phi(y)v \ , \quad y \in V.$$

Every rank one transformation takes this form and when the underlying space V is finite–dimensional (so that trace makes sense on $L(V)$), we have $\mathrm{tr}(v \otimes \phi) = \phi(v)$.

Definition 2.1. Let S be a linear subspace of $L(V)$ and suppose $b \in L(V)$.

(1) Given a positive integer k, we say that b belongs to the *k–reflexive closure* of S if for each sequence $v_1, \ldots, v_k$ in V, there is a transformation $a \in S$ satisfying $av_i = bv_i$ for $i = 1, \ldots, k$.
(2) b is said to belong to the *strict closure* of S if it belongs to the k–reflexive closure of S for each $k \in \mathbb{N}$.
(3) The k–reflexive and strict closures of S are denoted by $\mathrm{ref}_k\, S$ and $\mathrm{str}\, S$ respectively. S is said to be *k–reflexive* if $S = \mathrm{ref}_k\, S$.

Reference to k is suppressed when it is 1.

When $A \subset L(V)$ is an identity containing algebra, $b \in \mathrm{ref}\, A$ if and only if b leaves invariant each A–invariant subspace of V, i.e., $\mathrm{ref}\, A = \mathrm{alg\, lat}\, A$ in the usual notation. A useful consequence of this observation is that all "closure operations" of Definition 2.1 respect direct products (see the proof of Proposition 3.4 below).

For any S, we have the chain of inclusions

$$S \subset \mathrm{str}(S) \subset \cdots \subset \mathrm{ref}_3(S) \subset \mathrm{ref}_2(S) \subset \mathrm{ref}_1(S). \tag{2.1}$$

Example 2.2. Subject to the obvious restriction, it is possible to specify the positions of proper inclusion in Display 2.1 arbitrarily.

Proof. The obvious restriction is that if $\mathrm{ref}_{k+1}\, S = \mathrm{ref}_k\, S$ for all $k \geq k_0$ then $\mathrm{str}\, S = \mathrm{ref}_k\, S$ for all such k as well.

Taking S to be the space of all finite rank transformations on an infinite–dimensional V, we see that it is possible to arrange proper inclusion at the leftmost position of (2.1), with equality elsewhere.

Given $k \in \mathbb{N}$, take $V = F^{k+1}$ and fix an invertible transformation $b \in L(V)$ with non–zero trace. (The identity transformation will do unless $k + 1$ is a multiple of the characteristic of F.) Take

$$S \equiv \{a \in L(V) : \mathrm{tr}(ba) = 0\}.$$

We check that the identity transformation, e, belongs to $\mathrm{ref}_k\, S$. Indeed, given any set of k vectors in V, we can find a non–zero $\phi \in V^*$ which annihilates them. Next, apply invertibility of b to choose $v \in V$ with $\phi(bv) = \mathrm{tr}\, b$. Then $a \equiv e - v \otimes \phi$ belongs to S and agrees with e on the given set of k vectors.

Since S has codimension one in $L(V)$, we conclude that $\mathrm{ref}_k\, S = L(V)$. Since $\mathrm{ref}_{k+1}\, S = S$, we get proper inclusion at the k'th position from the right in (2.1), with equality elsewhere.

The proof is completed by taking various direct products of the concrete spaces we have constructed so far.

The time–honored trick of embedding linear spaces of transformations in algebras of upper–triangular block matrices (e.g. Proposition 3.9 of [1]) shows that arbitrariness in Display 2.1 persists even if we restrict attention to commutative algebras of transformations. The situation is quite different for the singly generated algebras which we will study in the balance of the paper.

3. Singly generated algebras

Definition 3.1. Fix $a \in L(V)$ and regard the underlying vector space V as a module over the polynomial ring $F[x]$.

(1) We write $\operatorname{pol} a$ for the subalgebra of $L(V)$ consisting of polynomials in a; we also write A for this algebra.
(2) The strict and reflexive closures of $\operatorname{pol} a$ are denoted by $\operatorname{str} a$ and $\operatorname{ref} a$ respectively.
(3) We write $\langle E \rangle$ for the submodule of V generated by the subset E of V. As usual, $\langle \{v\} \rangle$ is abbreviated to $\langle v \rangle$.
(4) The minimal polynomial of the restriction of a to $\langle E \rangle$ is denoted by p_E.

In other words, $\langle E \rangle$ is the smallest subspace of V which contains E and is invariant under a; in particular, $\langle v \rangle = Av$.

Part (1) of Theorem 3.3 below is due to L. Brickman and P. A. Fillmore [4]; part (3) is the result of Hadwin [10] mentioned in the Introduction. The proofs given here are essentially those of the original authors. The simple fact recorded in Proposition 3.2 is implicit in all proofs concerning reflexivity; fancier versions of this principle can be found in [7] and [8].

Proposition 3.2. *Let $a \in L(V)$ and suppose that $b \in \operatorname{ref} a$ satisfies $bv = 0$ for some $v \in V$. Then $bw = 0$ whenever $\langle v \rangle \cap \langle w \rangle = \{0\}$ and p_w divides p_v.*

Proof. Choose a polynomial q satisfying $b(v+w) = q(a)(v+w)$. Since $\langle v \rangle$ and $\langle w \rangle$ are invariant under a, we must have $0 = bv = q(a)v$ and $bw = q(a)w$. Thus p_v divides q, so p_w must also divide q whence $bw = 0$.

Theorem 3.3. *Let $a \in L(V)$ and write A for the subalgebra* $\operatorname{pol} a$ *of $L(V)$ consisting of polynomials in a.*

(1) $\operatorname{str} A = A' \cap \operatorname{ref} A$.
(2) *In particular, Display 2.1 collapses to*

$$A \subset \operatorname{str} A = \operatorname{ref}_2 A \subset \operatorname{ref} A. \tag{3.1}$$

(3) *If a is not locally algebraic, then $A = \operatorname{ref} A$.*

Proof.

(1) The opposite inclusion being obvious, suppose that $b \in A' \cap \operatorname{ref} A$, and let E be a finite subset of V. As a finitely generated module over a principal ideal domain, we can express $\langle E \rangle = \langle v_1 \rangle \oplus \cdots \oplus \langle v_n \rangle$ with the minimal

polynomial p_{v_1} of the restriction of a to $\langle v_1\rangle$ coinciding with the minimal polynomial p_E of $a|_{\langle E\rangle}$. Choose a polynomial q with $bv_1 = q(a)v_1$ and write $c = b - q(a)$. Then c annihilates v_1 by construction, whence c vanishes on $\langle v_1\rangle$ because it commutes with a. On the other hand, Proposition 3.2 implies that c vanishes on $\langle v_2\rangle, \ldots, \langle v_n\rangle$. We conclude that $b = q(a)$ on $\langle E\rangle$. The arbitrariness of E shows that $b \in \operatorname{str} A$ as desired.

(2) This follows from (1) and the obvious inclusion $\operatorname{ref}_2 A \subset A' \cap \operatorname{ref} A$.

(3) Choose a vector $v \in V$ which separates polynomials in the sense that $p(a)v = 0$ only when $p = 0$. Suppose that $b \in \operatorname{ref} A$ and E is a finite subset of V containing v. Then we can express $\langle E\rangle = \langle v_1\rangle \oplus \cdots \oplus \langle v_n\rangle$ with v_1 separating polynomials. Choose a polynomial q with $bv_1 = q(a)v_1$ and write $c = b - q(a)$. We have $cv_1 = 0$ by construction, so Proposition 3.2 tells us that c vanishes on $\langle v_2, \ldots, v_n\rangle$.

Necessarily $cav_1 = p(a)v_1$ for some polynomial p. Given a scalar λ, we have $p(a)v_1 = c(a - \lambda e)v_1$ whence $p(a)v_1 \in (a - \lambda e)Av_1$. Since v_1 separates polynomials, this implies that $x - \lambda$ divides p. Because the underlying field is infinite, we conclude $cav_1 = 0$, whence an induction argument implies that c vanishes on $\langle v_1\rangle$. Thus b agrees with $q(a)$ on $\langle E\rangle$. Recalling that the vector $v \in E$ separates polynomials, we see that q is independent of E, so $b \in A$ as desired.

The following result, basically Theorem 3 of [6], adapts the primary decomposition of [5] to vector spaces of arbitrary dimension.

Proposition 3.4. *Suppose $a \in L(V)$ is locally algebraic and write A for the subalgebra of $L(V)$ it generates. For each irreducible polynomial $p \in F[x]$, set*

$$V_p = \bigcup_{n\in\mathbb{N}} \ker p^n(a).$$

Write a_p for the restriction of a to V_p and A_p for the algebra generated by a_p.

1. $V = \bigoplus_{p \text{ irr}} V_p$.
2. $\operatorname{ref} A = \prod_{p \text{ irr}} \operatorname{ref} A_p$.
3. $\operatorname{str} A = \prod_{p \text{ irr}} \operatorname{str} A_p$.
4. *In particular,* $\operatorname{str} A$ *is reflexive if and only if* $\operatorname{str} A_p$ *is reflexive for each irreducible p.*

Proof. We make the usual distinction between direct sums and direct products. Thus (1) means that the $\{V_p : p \text{ irreducible}\}$ are mutually independent and span V; these are consequences of the Chinese Remainder Theorem for polynomials. Since each V_p is invariant under A, each *individual* member $b \in \operatorname{ref} A$ is decomposable in the sense that $bv = \sum b_p v_p$ for each vector $v \in V$. Whether one thinks of this as a direct sum (reflecting the structure of V) or a direct product (reflecting the fact that b_p may fail to vanish for any p) is a matter of taste; we take the former course below. On the other hand, since A is a linear space, it would be wrong to think of A as a subset of the direct *sum* $\oplus A_p$. Thus the most that can be said in general is $A \subset \prod A_p$ and of course this inclusion can be quite proper. In any case, the $\subset$ inclusions of (2) and (3) follow from Definition 2.1.

To establish the reverse inclusion for (2), suppose $\oplus b_p \in \prod \operatorname{ref} A_p$. Given a vector $v \in V$, choose a non–zero polynomial q satisfying $q(a)v = 0$ and write $q = \prod_{i=1}^n p_i^{k_i}$ for its prime factorization. Then we can write $v = \sum_{i=1}^n v_i$ where v_i belongs to $\ker p_i^{k_i} \subset V_{p_i}$. By definition of reflexive closure, for each i, there is a polynomial q_i such that $q_i(a)$ agrees with b_{p_i} on v_i. Apply the Chinese Remainder Theorem to get a polynomial r simultaneously satisfying $r \equiv q_i \mod p_i^{k_i}$ for $i = 1, \ldots, n$. Thus $r(a)$ agrees with $\oplus b_p$ on v, and we have shown that $\prod \operatorname{ref} A_p \subset \operatorname{ref} A$.

Replacing ref by ref_2 and v by a pair of vectors v, w in the preceding paragraph, we conclude that $\operatorname{ref}_2 A = \prod \operatorname{ref}_2 A_p$, whence (3) follows by Theorem 3.3(2).

Part (4) is a direct consequence of (2) and (3).

The next result (from [10]) settles the question of when the left containment in Display 3.1 is proper. Following Delaï, we therefore focus on the right containment in the basic Definition 3.6.

Corollary 3.5. *If a is locally algebraic but not algebraic, then A is properly contained in* $\operatorname{str} A$*; otherwise they are equal.*

Proof. If a is not locally algebraic, then $A = \operatorname{str} A$ by Theorem 3.3(3); since every finite–dimensional subspace of $L(V)$ is strictly closed, the same conclusion holds if a is algebraic.

Conversely, suppose that a is locally algebraic and $A = \operatorname{str} A$. For each irreducible p, the operator $p(a_p)$ is locally nilpotent on V_p, so the infinite series $\sum_{i \in \mathbb{N}} p^i(a_p)$ belongs to $\operatorname{str} A_p$. Since $A_p = \operatorname{str} A_p$, this series must in fact be finite, and we see that each a_p is algebraic. Since A has countable dimension, the equation $A = \operatorname{str} A = \prod \operatorname{str} A_p$ implies that V_p vanishes for all but finitely many p. Thus $a = \oplus a_p$ is algebraic.

Definition 3.6 (Delaï). An individual linear transformation $a \in L(V)$ is *reflexive* if $\operatorname{str} a = \operatorname{ref} a$.

We conclude this section with a brief comparison of our purely algebraic setting with its topological analogue. For definiteness, let H be a Hilbert space. Attention is restricted to *closed* subspaces of H and *bounded* operators on H. Thus the definition of reflexive closure becomes

$$\operatorname{ref} S = \{b \in B(H) : \ bx \in \overline{Sx} \quad \text{for each } x \in H\}$$

where the bar indicates closure in the norm topology. The intersection $\cap_{k \in \mathbb{N}} \operatorname{ref}_k S$ is the closure of S in the *strong operator topology* on $B(H)$; this is denoted by $\operatorname{sot} S$. Display 2.1 remains valid in the Hilbert space setting; actually it can be lengthened by considering various other topologies on $B(H)$.

The Baire Category Theorem shows that if $a \in B(H)$ is non–algebraic, then $\operatorname{pol} a$ can not even be uniformly closed. In particular, there is no chance of having $\operatorname{pol} a$ reflexive for such operators. The universal convention is thus to call an individual operator $a \in B(H)$ *reflexive* if the **sot**–closed algebra generated by a is reflexive. Similar reasoning applies to the framing of Definition 3.6.

Theorem 3.3 says that singly generated subalgebras of $L(V)$ come close to being reflexive. The analoguous statement fails rather spectacularly in the Hilbert space setting. In [14], W. Wogen showed how to embed arbitrary subspaces of $B(H)$ in singly generated operator algebras, and he applied this technique to provide counterexamples to Parts (1) and (2). In fact [2], points of proper inclusion in Display 2.1 can still be arbitraily specified for singly generated subalgebras of $B(H)$.

As for Theorem 3.3(3), a category argument (Theorem 15 of [12]) shows that every locally algebraic operator in $B(H)$ must be algebraic. In particular, the complementary subset of $B(H)$ has many non–reflexive members—we are thrilled to find new classes of them with any non-trivial invariant subspaces at all.

4. Primary transformations

We return to the algebraic setting where the underlying vector space V is not equipped with any topology. Following Kaplansky [12], a transformation $a \in L(V)$ is called *primary* if $p(a)$ is locally nilpotent for some irreducible polynomial p. The transformation $a = 0$ is allowed, for which we take $p(x) = x$. Proposition 3.4 reduces the study of locally algebraic transformations to the primary ones and we concentrate on them in this section.

Answers to concrete questions should be phrased in terms of "simple" quantities. In this section, we discuss a sequence of "computable" numbers associated with primary transformations. Corollary 4.7 provides the bridge between these numbers and the concrete description of the rank one members of $\operatorname{ref} a$ of Theorem 5.4. The prepartory material in 4.1–4.6 adapts various classical results to our purposes.

Definition 4.1. The *block invariants* of a primary transformation $a \in L(V)$ are defined by

$$\mathcal{B}_k(a) = \dim\ [\ \ker p(a) \cap \operatorname{ran} p^k(a)\ ], \quad k \in \mathbb{N}.$$

The sequence $\{\mathcal{B}_k(a)\}$ is clearly non–increasing, and invariant under similarity.

Example 4.2. All operators in this example act on finite–dimensional spaces.

Write s_n for the (backward) shift acting on F^{n+1}. Then we have $\mathcal{B}_k(s_n) = 1$ for $k \le n$ and $\mathcal{B}_k(s_n) = 0$ otherwise.

Since dimensions, kernels, and ranges respect direct sums, we see that $\mathcal{B}_k$ counts the number of blocks of size at least $k+1$ in the Jordan Canonical Form of any nilpotent transformation.

More generally, if $p(a)$ is nilpotent for some irreducible polynomial p, then $\mathcal{B}_k(a)$ can be computed by counting the blocks in the Classical Canonical Form of a having size at least $(\deg p)(k+1)$ and multiplying the result by $\deg p$.

In particular, block invariants form a complete set of similarity invariants for primary transformations acting on finite–dimensional spaces. They do not, however, distinguish between $\oplus s_n$ and $\oplus s_{2n}$. For such tasks, one needs to consider dimensions of quotients of the spaces appearing in Definition 4.1 and we postpone such considerations to Section 6.

Proposition 4.3. *Suppose $a \in L(V)$ is primary and $k \in \mathbb{N}$.*

1. *Each finite block number is divisible by the degree of p.*
2. *$\mathcal{B}_k(a)$ is the dimension of the quotient space $\ker p^{k+1}(a)/\ker p^k(a)$.*
3. *In order that $A = \operatorname{pol} a$ act transitively on $\ker p^{k+1}(a)/\ker p^k(a)$, it is nececssary and sufficient that $\mathcal{B}_k(a) \le \deg p$.*

Proof. Write K for the field $F[x]/pF[x]$. We make $\ker a$ into a vector space over K by defining $\overline{f}v = f(a)v$ for each $f \in F[x]$ and $v \in V$. By this definition, an F–closed subsubspace of $\ker a$ is invariant under a if and only if it is closed under muliplication by K. Thus (1) follows from the fact that every a–invariant subspace M of $\ker a$ satisfies $\dim M = (\deg p)(\dim_K M)$. (Unqualified references to dimension are taken with respect to F).

Since $p^k(a)$ maps $\ker p^{k+1}(a)$ onto $\ker p(a) \cap \operatorname{ran} p^k(a)$, the first isomorphism theorem tells us that $\ker p(a) \cap \operatorname{ran} p^k(a)$ has the same dimension (over F) as the quotient space $Q \equiv \ker p^{k+1}(a)/\ker p^k(a)$. This serves to establish (2).

Finally, we get (3) by making the quotient space Q into a vector space over K, noting that K acts transitively on Q iff $\dim_K Q = 1$.

The *order* of a vector $v \in V$ is the smallest $n \in \mathbb{N}$ satisfying $p^n(a)v = 0$; when $a = 0$, the order of the zero vector is taken to be zero, but the order of every other vector is taken to be one. (This usage conflicts with [12] where the order of a vector refers to the minimal polynomial annihilating it.)

Proposition 4.4. *Suppose a is primary and* $\dim[\ker p(a) \cap \operatorname{ran} p^k(a)] > \deg p$. *Then given v of order $k+1$, there is a vector w of order $k+1$ with $Av \cap Aw = \{0\}$.*

Proof. Write $Q \equiv \ker p^{k+1}(a)/\ker p^k(a)$. In view of Proposition 4.3, the hypothesis means $\dim_K Q > 1$ so the proof is completed by choosing $w \in \ker p^{k+1}(a)$ such that $w + \ker p^k(a)$ is independent of $v + \ker p^k(a)$ over K.

Proposition 4.5. *Let $a \in L(V)$ be primary. Then the following are equivalent.*

1. $\dim[\ker p(a)] \leq \deg p$.
2. $Av = \ker p^{\operatorname{ord} v}(a)$ *for some non–zero vector $v \in V$.*
3. $Av = \ker p^{\operatorname{ord} v}(a)$ *for every vector $v \in V$.*
4. *The invariant subspace lattice of a is totally ordered by inclusion.*

Proof. We leave it to the reader to check that the conventions have been arranged to make these conditions equivalent for the zero operator. We restrict attention to non–zero a in the rest of the proof.

Assuming (1), Proposition 4.3 tells us that A acts transitively on $\ker p(a)$, so (2) holds.

Assume (2) so that $Aw = \ker p^k(a)$ for some vector w of order $k > 0$. Given $y \in \ker p(a)$, there must be some polynomial f satisfying $f(a)v = y$. Comparing orders, we see that f is divisible by p^{k-1} whence $Ap^{k-1}(a)w = \ker a$, i.e., we may as well assume that the original vector w has order one. In particular, we see that $\dim \ker p(a) = \dim Aw = \deg p$ so (1) and (2) are equivalent.

We now establish (1) $\implies$ (3) by arguing inductively on the order of v. There is nothing to do for order zero. Given v of order $n+1$, we apply the inductive hypothesis to $p(a)v$, concluding that $\ker p^n(a) \subset Av$. But $\mathcal{B}_n(a) \leq \mathcal{B}_0(a)$ so A acts transitively on $\ker p^{n+1}(a)/\ker p^n(a)$. In other words Av contains representatives of each coset in this quotient space and we have $\ker p^{n+1}(a) \subset Av$. Since the opposite inclusion is automatic, we have completed the inductive argument.

(3) means that every proper invariant subspace for a takes the form $\ker p^k(a)$ and since these are totally ordered by inclusion, we get (3) $\implies$ (4).

Finally (4) forces the invariant subspaces of $\ker p(a)$ to be totally ordered by inclusion, which means $\dim_K \ker p(a) \leq 1$ so (4) $\implies$ (1) and the proof is complete.

Example 4.6. When V is finite dimensional, one can add two additional conditions to the preceding proposition:

(5) a admits a cyclic vector.

(6) The canonical form of a has a single block.

On infinite–dimensional spaces, however, no locally algebraic transformation can satisfy (5). On the other hand, there is a backward shift on such a space satisfying (1) thru (4). More precisely, let F^ω denote the vector space of F–valued sequences having finite support, and define s_ω by

$$(S_\omega v)_n = v_{n+1}, \qquad v \in F^\omega, \quad n \in \mathbb{N}.$$

As we will see in Section 6, the appropriate version of (6) for infinite–dimensional V is "*a admits a canonical form* and that form has a single block".

The next result reduces to Proposition 4.4 when $n \leq k$ and to Proposition 4.5 when $k = 0$.

Corollary 4.7. *Suppose a is primary and interpret $p^\infty(a)$ as the zero operator. Let k be the smallest member of $\mathbb{N} \cup \{\infty\}$ for which $\dim[\ker p(a) \cap \operatorname{ran} p^k(a)] \leq \deg p$. For each $n \in \mathbb{N}$ we have*

$$\bigcap_{\operatorname{ord} y = n} Ay = p^k(a) \ker p^n(a). \tag{4.1}$$

Proof. Fix n and write M for the intersection appearing in Display 4.1.

If $k = \infty$, Proposition 4.4 yields vectors v, w of order n with $Av \cap Aw = \{0\}$ so $M = \{0\}$ as desired.

Suppose now that $k < \infty$. Given z of order n, the set of polynomials q for which $q(a)z \in M$ is an ideal in $F[x]$ so $M = \langle p^m(a)z \rangle$ for some integer $m \leq n$. In particular, $\dim M = (n - m)d$, so m is in fact indepedent of z whence $M = p^m(a) \ker p^n(a)$.

It remains to show that $m = k$. Fix $v \in V$ of order n. Set $\hat{V} = \operatorname{ran} p^k(a)$, write $\hat{a}$ for the restriction of a to $\hat{V}$, and take $\hat{A} = \operatorname{pol} \hat{a}$. Then the vector $p^k(a)v$ will have order $n - k$ as a member of $\hat{V}$ so Proposition 4.5 yields $\hat{A}p^k(a)v = \ker p^{n-k}(\hat{a})$. Translating back to V, this implies

$$Ap^k(a)v = \ker p^{n-k}(a) \cap \operatorname{ran} p^k(a) = p^k(a) \ker p^n(a).$$

In particular, this implies that $Av \supset p^k(a) \ker p^n(a)$ so $m \leq k$. This completes the proof if $k = 0$.

On the other hand, if $k > 0$, we know that $\mathcal{B}_{k-1}(a) > \deg p$. Applying Proposition 4.5 to the restriction of a to $\operatorname{ran} p^{k-1}(a)$ we then conclude

$$Ap^{k-1}(a)v \not\supseteq p^{k-1}(a) \ker p^n(a). \tag{4.2}$$

Now if $f(a)v$ belongs to the right hand side of Display 4.2, its order cannot exceed $n - (k - 1)$ and thus f must be divisible by p^{k-1}. Thus Display 4.2 is equivalent to

$$Av \not\supseteq p^{k-1}(a) \ker p^n(a)$$

whence $m > k - 1$ and the proof is complete.

Example 4.8. We investigate the meaning of Corollary 4.7 for a nilpotent transformation a acting on a finite–dimensional space. Suppose a is in (upper–triangular) Jordan canonical form relative to the standard basis $e_1, e_2, \dots,$ with block sizes arranged in non–increasing order. Write $m_1 \geq m_2$ for the sizes of the two largest blocks. ($m_2 = 0$ if there is only one block.) We have $p(x) = x$ and $k = m_2$. Thus Display 4.1 becomes

$$\bigcap_{\operatorname{ord} y = n} Ay = \begin{cases} \operatorname{span}\{e_1, \dots, e_{n-m_2}\}, & \text{if } m_2 < n \leq m_1 \\ \{0\}, & \text{otherwise.} \end{cases} \tag{4.2}$$

In particular, for each n, we have

$\cap\{Ay : \operatorname{ord} y = n\} = \{0\}$ when $m_2 = m_1$ and
$\cap\{Ay : \operatorname{ord} y = n\} = \ker a^n$ when $m_2 = 0$.

These are consistent with Propositions 4.4 and 4.5 respectively.

5. Transformations of rank one

Throughout this section, A denotes the algebra generated by a transformation $a \in L(V)$. In view of Theorem 3.3(3) and Proposition 3.4, we concentrate on primary a. After finding the rank one members of $\operatorname{str} a$ in Proposition 5.3, we characterize the rank one members of $\operatorname{ref} a$ in Theorem 5.4. The pieces are then assembled in Theorem 5.8 which states that any excess of $\operatorname{str} a$ over $\operatorname{ref} a$ must be due to transformations of rank one.

The pioneering result concerning reflexive transformations is due to J. A. Deddens and P. A. Fillmore [5]; this can be paraphrased as follows.

Proposition 5.1. *Suppose $a \in L(V)$ is a nilpotent transformation acting on a finite–dimensional space and write $m_1 \geq m_2$ for the sizes of the two largest blocks in its Jordan Form. Then a is reflexive if and only if either $m_2 = m_1$ or $m_2 = m_1 - 1$.*

Successive generalizations in [10] and [6] allow $p(a)$ to be locally nilpotent for any irreducible polynomial p, remove the dimensionality restriction on V, and finally allow $p(a)$ to be locally nilpotent. An important distinction between [5] and its successors is that [5] catalogues all members of $\operatorname{ref} a$ regardless of whether a is reflexive. The goal of the present section is a unified analysis of this type for all members of $L(V)$. Our main result states that the excess of $\operatorname{ref} a$ over $\operatorname{pol} a$ can always be accounted for by rank one operators.

Proposition 5.2. *Suppose a is primary and consider the formal power series*

$$\sum_{i=0}^{\infty} q_i(a)p^i(a), \quad \deg q_i < \deg p, \quad i \in \mathbb{N}. \tag{5.1}$$

(1) *Every series of the form (5.1) defines a member of* $\operatorname{str} a$, *and every member of* $\operatorname{str} a$ *admits such a representation.*
(2) *The series (5.1) represents the zero operator if and only if each of its terms is zero.*

Proof. The first assertion of (1) is clear since each finite–dimensional subspace of V is annihilated by a power of $p(a)$.

Conversely, given $b \in \operatorname{str} a$ and $v \in V$ of order n, there is a polynomial of the form $\sum_{i=0}^{n-1} q_i p^i$ whose value at a agrees with b on v. By requiring the degrees of the $\{q_i\}$ to be smaller than d, we guarantee uniqueness of the polynomial associated with the given vector v.

To see that the $\{q_i\}$ are independent of v, suppose $w \in V$ has order $m \geq n$ and write $\sum_{i=0}^{m-1} q'_i p^i$ for the associated polynomial. Since $b \in \operatorname{str} a$, there is a polynomial r simultaneously satisfying $bv = r(a)v$ and $bw = r(a)w$. Then we must have

$$r \equiv \sum_{i=0}^{n-1} q_i p^i \mod p^n\,, \qquad r \equiv \sum_{i=0}^{m-1} q'_i p^i \mod p^m\,,$$

whence $\sum_{i=0}^{n-1} q_i p^i \equiv \sum_{i=0}^{n-1} q'_i p^i \mod p^n$. This forces $q_i = q'_i$ for $i < n$. Thus the $\{q_i\}$ associated with different vectors are consistent, and they can be assembled into a single series of the form (5.1).

For (2), let $q_n(a)p^n(a)$ be the lowest degree non–vanishing term in (5.1). Then V contains vectors of order $n+1$ and such vectors will not lie in the kernel of (5.1).

Proposition 5.3. *For primary a, the only possible rank one members of* $\operatorname{str} a$ *are scalar multiples of $p^{n-1}(a)$ where $p(a)$ is nilpotent of order n. In particular,* $\operatorname{str} a$ *can only have rank one members if p is linear and a is algebraic.*

Proof. Let $b = \sum q_i(a)p^i(a)$ be a rank one member of $\operatorname{str} a$. Since zero is the only eignevalue of $p(a)$, we see that $p(a)b = 0$; in view of Proposition 5.2, b has a single non–zero term, $q_{n-1}(a)p^{n-1}(a)$, with $p^n(a) = 0$ and $\deg q_{n-1} < \deg p$. Moreover, if v is a non–zero vector in the range of b, then v, av must be dependent, which makes p of first degree and q_{n-1} constant.

We now proceed to describe the rank one members of $\operatorname{ref} a$ in terms of its block numbers.

Theorem 5.4. *Suppose a is primary, $v \otimes \phi$ is a rank one member of $L(V)$, and n is the smallest possible order for vectors not belonging to* $\ker \phi$. *Then the following are equivalent.*

(1) $v \otimes \phi \in \operatorname{ref} a$.
(2) $v \in Aw$ *for each w not belonging to* $\ker \phi$.
(3) $v \in Aw$ *for each w of order n.*
(4) $v \in p^k(a) \ker p^n(a)$ *for some $k < n$ with* $\dim[\ker p(a) \cap \operatorname{ran} p^k(a)] = \deg p$.

Proof. We begin by noting that there are non–zero vectors in $\ker p(a) \cap \operatorname{ran} p^{n-1}(a)$, so (4) $\iff$ (3) is the content of Corollary 4.7.

Assume next that (3) is satisfied and let $y \in V$. If the order of y is less than n, we have $(v \otimes \phi)y = 0 \in Ay$. On the other hand, if the order of y is at least n, then there is a polynomial f for which $f(a)y$ has order equal to n whence $(v \otimes \phi)y \in Af(a)y \subset Ay$. This establishes (3) $\Longrightarrow$ (1).

Suppose (1) holds and $\phi(w) \neq 0$. Then $(v \otimes \phi)w$ is a non–zero scalar multiple of v and it must belong to Aw. Thus (1) $\Longrightarrow$ (2).

We complete the proof by establishing (2) $\implies$ (3). Assume (2) and fix z of order n with $\phi(z) = 1$. We consider two cases. If $\dim[p^{n-1}(a) \ker p^n(a)] = d$, then Proposition 4.3 tells us that A acts transitively on $\ker p^n(a)/\ker p^{n-1}(a)$. Thus given w of order n, we can write $z = f(a)w + y$ for some $y \in \ker p^{n-1}(a)$. Since $\phi(z - y) = 1$ this yields $v \in A(z - y) \subset Aw$, as required by (3).

The remaining possibility is $\dim[p^{n-1}(a) \ker p^n(a)] > d$. Here, Proposition 4.4 yields a vector w of order n with $Aw \cap Az = \{0\}$. We can in fact arrange $\phi(w) \neq 0$. Indeed if $\phi(w) = 0$, replace w by $w + \lambda z$ where the non–zero scalar λ is chosen so that the latter vector still has order n. But this means (2) is not satisfied by any non–zero vector v so this case does not actually arise.

Example 5.5. We informally investigate the meaning of Theorem 5.4 for a nilpotent transformation a acting on a finite–dimensional space. Continuing with the notation of Example 4.8, assume a is in (upper–triangular) Jordan canonical form relative to the standard basis $e_1, e_2, \ldots$, with block sizes arranged in non–increasing order. Write $m_1 \geq m_2$ for the sizes of the two largest blocks.

Suppose $v \otimes \phi$ is a rank one member of $\operatorname{ref} a$ and n is as in the statement of Theorem 5.4. The presence of vectors of order n in V forces $m_1 \geq n$. Because ϕ kills all vectors of smaller order, it must be supported on the span of $e_n \ldots e_{m_1}$. On the other hand, from Example 4.8, we know that Condition (4) is equivalent to having $v \in \operatorname{span}\{e_1 \ldots e_{n-m_2}\}$.

Thus (4) is equivalent to demanding that $v \otimes \phi$ be supported on rows $1 \ldots n - m_2$ and columns $n \ldots m_1$.

We note some special cases.

1. If $m_2 = 0$ (only one block), then (4) reduces to the requirement that $v \otimes \phi$ be upper–triangular.
2. At the other extreme, if $m_2 = m_1$, then $\operatorname{ref} a$ has no rank one members.
3. If $m_2 = m_1 - 1$, the rank one members of $\operatorname{ref} a$ are supported on the single position $(1, m_1)$.
4. If $m_2 < m_1 - 1$ then $\operatorname{ref} a$ contains an independent pair of rank one members.

In particular, the condition $m_2 \geq m_1 - 1$ is equivalent to having the rank one members of $\operatorname{ref} a$ and $\operatorname{str} a$ coincide. This reveals Proposition 5.1 as a special case of Theorem 5.8 below.

Lemma 5.6. *Suppose the order of x does not exceed the order of v and write k for the smallest integer satisfying $p^k(a)x \in \langle v \rangle$. Then $\langle v, x \rangle = \langle v \rangle \oplus \langle y \rangle$ for some vector y of order k.*

Proof. Write $p^k(a)x = f(a)p^l(a)v$ with f relatively prime to p. Comparing orders, we see that $l \geq k$. Set $y = x - f(a)p^{l-k}(a)v$. Clearly, $\langle v, x \rangle = \langle v, y \rangle$. Since the order of y is k, we also have $\langle v \rangle \cap \langle y \rangle = \{0\}$ as desired.

Proposition 5.7. *Given a primary, $b \in \operatorname{ref} a$, and $n \in \mathbb{N}$, there is a polynomial f and a finite linear combination c of rank one members of $\operatorname{ref} a$ such that b agrees with $f(a) + c$ on $\ker p^n(a)$.*

Proof. We argue inductively on n. There is nothing to do for $n = 0$. Assuming we can implement the construction on $\ker p^n(a)$, we show how to adapt the decomposition to $\ker p^{n+1}(a)$. The procedure depends on $\ell \equiv \dim[\ker a \cap \operatorname{ran} p^n(a)]$. If $\ell = 0$, then $\ker p^{n+1}(a) = \ker p^n(a)$ and no adjustment is necessary.

Assume next that $\ell > d$ and apply Proposition 4.4 to find vectors v, w of order $n+1$ with $\langle v \rangle \cap \langle w \rangle = \{0\}$. By definition of ref, there is a polynomial g such that $b - g(a)$ vanishes on v. Now let x be an arbitrary member of $\ker p^{n+1}(a)$ and apply Lemma 5.6 to express $\langle v, x \rangle = \langle v \rangle \oplus \langle y \rangle$ for some y of order at most $n+1$. We now appeal to Proposition 3.2 three times to conclude that $b - g(a)$ vanishes on w, $\langle v \rangle$, and $\langle y \rangle$ respectively. In particular, $bx = g(a)x$, and we have shown that b agrees with $g(a)$ throughout $\ker p^{n+1}(a)$.

It remains to consider the case $\ell = d$. We begin by invoking the inductive hypothesis to find a polynomial g and a finite linear combination c of rank one members of $\operatorname{ref} a$ such that $r \equiv b - c - g(a)$ vanishes on $\ker p^n(a)$. Next apply the dimensionality assumption to find vectors $v_1, \ldots, v_d$ of order $n+1$ such that the $\{v_i + \ker p^n(a) : i \leq d\}$ form a basis for the quotient space $\ker p^{n+1}(a)/\ker p^n(a)$. Then we choose a dual set in V^*, that is functionals $\phi_1, \ldots, \phi_d$ which vanish on $\ker p^n(a)$ and satisfy $\phi_i(v_j) = \delta_{i,j}$ for $i, j \leq d$.

Fix i for the moment, and suppose w has order $n+1$. Our dimensionality assumption means that A acts transitively on $\ker p^{n+1}(a)/\ker p^n(a)$ so we can find a polynomial f with $v_i - f(a)w \in \ker p^n(a)$. Since $r \in \operatorname{ref} A$, this yields

$$rv_i = rf(a)w \in Af(a)w \subset Aw. \tag{5.2}$$

Applying the equivalence (1) $\iff$ (3) of Theorem 5.4, we therefore conclude that each $rv_i \otimes \phi_i \in \operatorname{ref} A$. The proof is thus completed by observing that $r - \sum_{i=1}^d rv_i \otimes \phi_i$ vanishes on $\ker p^{n+1}$ whence b agrees with $g(a) + [c + \sum_{i=1}^d rv_i \otimes \phi_i]$ on $\ker p^{n+1}(a)$.

Theorem 5.8. *Let A be the algebra generated by a single linear transformation. Then $\operatorname{ref} A$ is the strict closure of the span of those of its members which either have rank one or belong to A.*

Proof. In view of Theorem 3.3(3), we may assume the generator a of A to be locally algebraic; applying Propostion 3.4, we may also assume a to be primary. The proof is therefore completed by appealing to Proposition 5.7.

In the following Corollary, we recover the characterizations of reflexive transformations discovered by Hadwin [10] and Delaï [6]. From the perspective of the present paper, the presence of Condition (2) is explained by Proposition 5.3—it is only when p is of first degree and a is algebraic that $\operatorname{str} a$ has a chance of containing any rank one members.

Recall the block numbers from Section 4:

$$\mathcal{B}_k(a) \equiv \dim \left[\ \ker p(a) \cap \operatorname{ran} p^k(a) \ \right] = \dim \ \frac{\ker p^{k+1}(a)}{\ker p^k(a)}.$$

Corollary 5.9. *Suppose an operator a on V has the property that $p(a)$ is locally nilpotent for the irreducible polynomial p. Then a is reflexive if and only if either*

1. *$\mathcal{B}_k(a)$ does not agree with the degree of p for any integer k, or*
2. *p is of first degree and $\mathcal{B}_k(a) = 1$ for a unique value of k.*

Proof. In view of Theorem 5.8, we know that $\operatorname{str} A = \operatorname{ref} A$ precisely when all rank one members of $\operatorname{ref} A$ belong to $\operatorname{str} A$. If $\deg p > 1$, Proposition 5.3 tells us that str A has no rank one members, while Theorem 5.4 tells us that the same is true of ref A iff the condition of (1) is satisfied.

Thus we may as well assume that p is of first degree. If $\mathcal{B}_k(a)$ is never one, then str A = ref A as in the preceding paragraph. On the other hand, if $\mathcal{B}_k(a) = 1$ for more than one value of k, then Theorem 5.4 guarantees an independent pair of rank one members in ref A, so Proposition 5.2 precludes equality of str A and ref A.

Suppose finally that $\mathcal{B}_k(a) = 1$ for a unique integer k. Then $\mathcal{B}_{k+1}(a) = 0$, so $p^{k+1}(a) = 0$ and $p^k(a)$ has rank one. Given a rank one member $v \otimes \phi$ of ref A, Theorem 5.4 yields $\ker p^k(a) \subset \ker \phi$, whence $v \otimes \phi$ is a scalar multiple of $p^k(a)$. Thus all rank one members of ref A belong to A, and str A = ref A by Theorem 5.8.

To see that Proposition 5.1 is a special case of this result, note that the condition $m_2 \geq m_1 - 1$ appearing there is equivalent to requiring $\mathcal{B}_k(a) = 1$ for *at most* one value of k. Several alternate ways of expressing Corollary 5.9 in the nilpotent case can be found in Lemma 2.5 of Hadwin and Nordgren's paper [11]; in fact, it was our reading of [11] which first led to the considerations of the present paper.

6. Reflexivity versus canonical forms

In this section, we apply ideas from I. Kaplansky's monograph [12] to explain the title of this paper. For simplicity of notation, we treat locally nilpotent transformations; the discussion is easily adapted to the setting of primary transformations.

Definition 6.1. A locally algebraic operator in $L(V)$ is *unicellular* if its lattice of invariant subspaces is totally ordered by inclusion. An operator is said to *admit a canonical form* if it is similar to some (possibly infinite) direct sum of unicellular operators.

Example 6.2. Recall the backward shifts s_n acting on F^{n+1} and s_ω acting on F^ω studied in Examples 4.2 and 4.6 respectively. These have one–dimensional kernels, so Proposition 4.5 tells us they are unicellular.

In the preceding section, we saw that block numbers determine whether a locally nilpotent operator is reflexive. These simple similarity invariants do not distinguish between the operators $\oplus_{n\in\mathbb{N}} s_n$ and $\oplus_{n\in\mathbb{N}} s_{2n}$. To do that, we consider dimensions of quotient spaces; these appear in Display 6.1 below.

Given an operator $b \in L(V)$ and a cardinal number n, we write $b^{(n)}$ for the direct sum of n copies of b. This operator can also be realized as the tensor product of b with the identity operator acting on an n–dimensional space and is usually referred to as the *n–fold ampliation* of b.

Proposition 6.3. *Suppose $a \in L(V)$ is locally nilpotent and admits a canonical form.*

(1) *If a is unicellular, it must be similar to one of the operators from Example 6.2.*
(2) *a is similar to a unique operator of the form $\bigoplus_{0\leq k\leq\omega} s_k^{(n_k)}$.*
(3) *a maps $\cap_{k\in\mathbb{N}} \operatorname{ran} a^k$ onto itself and n_ω is the dimension of the intersection of this space with* $\ker a$.

(4) *For each finite k, the cardinal number n_k equals*

$$\dim \left[\frac{\ker a \cap \operatorname{ran} a^k}{\ker a \cap \operatorname{ran} a^{k+1}} \right]. \tag{6.1}$$

Proof. For (1), suppose a is unicellular. Any non–zero vector $e_1 \in \ker a$ is a basis for $\ker a$. We need to extend this to a (finite or countable) basis $\{e_n\}$ for V satisfying $ae_{i+1} = e_i$ for all i. Assume such a basis $e_1, \dots, e_n$ for $\ker a^n$ has been constructed. If $\ker a^{n+1} = \ker a^n$, we're done. Otherwise, Proposition 4.5 tells us that $\ker a^n$ has codimension one in $\ker a^{n+1}$. In fact, the rank–nullity theorem tells us that a maps the latter space *onto* the former, so we can choose e_{n+1} satisfying $ae_{n+1} = e_n$ to get the desired basis $e_1, \dots, e_{n+1}$ for $\ker a^{n+1}$.

Existence in (2) is a matter of gathering similar direct summands. (3) and (4) are easily verified for the operators s_n, s_ω of Example 6.2; they extend to arbitrary direct sums of such operators because ranges, kernels and dimensions respect direct sums of subspaces. Finally, once (3) and (4) are established, they yield uniqueness in (2).

Example 6.4. There is a locally nilpotent operator which is reflexive, but does not admit a canonical form.

Construction. Take W to be the space of all lower–triangular infinite matrices having finite support and write $\{e_{i,j}\}_{i \geq j}$ for its standard basis. Let b act on W by "shifting one column to the left", i.e.,

$$be_{i,1} = 0, \quad be_{i,j} = e_{i,j-1} \qquad \text{for } i, j \in \mathbb{N}, \quad j > 1.$$

(b is similar to the transformation $\oplus_{n \in \mathbb{N}} s_n$.)

Now take M to be the set of matrices in $\ker b$ whose non–zero entries sum to zero, i.e.,

$$M = \{\sum_i c_i e_{i,1} : \ c_i = 0 \text{ for all but finitely many } i; \ \ \sum c_i = 0\}.$$

Then M is invariant under b, so b induces a locally nilpotent transformation a on the quotient space $V = W/M$.

Given n, note that

$$e_{1,1} + M = e_{n+1,1} + M = a^n(e_{n+1,n+1} + M) \quad \text{and}$$
$$e_{n+2,2} - e_{n+3,2} + M = a^n(e_{n+2,n+2} - e_{n+3,n+2} + M)$$

are independent members of $\ker a \cap \operatorname{ran} a^n$. Thus none of these spaces has dimension one and reflexivity of a follows from Corollary 5.9.

We next compute $\cap_{n \in \mathbb{N}} \operatorname{ran} a^n$. We already know that $e_{1,1} + M$ belongs to this space. On the other hand, $\operatorname{ran} b^n = \operatorname{span}\{e_{i,j} : i \geq j + n\}$ so $\cap_{n \in \mathbb{N}} [M + \operatorname{ran} b^n] = \operatorname{span}\{e_{i,1} : i \in \mathbb{N}\} = M + \operatorname{span} e_{1,1}$. This means that $\cap_{n \in \mathbb{N}} \operatorname{ran} a^n$ is the one–dimensional space spanned by $e_{1,1} + M$. Since a does not map this space onto itself, we see that Proposition 6.3(3) is violated whence a does not admit a canonical form.

The main point of the present section is that the behavior of Example 6.4 is typical. The reader familiar with Kaplansky's monograph [12] will recognize Proposition 6.6 below as a consequence of his Theorems 6, 2, and 4.

Lemma 6.5. *Suppose $a \in L(V)$, and $n, k \in \mathbb{N}$. If $\ker a \cap \operatorname{ran} a^k = \ker a \cap \operatorname{ran} a^{n+k}$, then $\ker a^n \cap \operatorname{ran} a^k \subset \operatorname{ran} a^{k+1}$.*

Proof. There is nothing to do when $n = 0$. When $n > 0$, the hypothesis implies that the transformation a^{n-1} maps $\ker a^n \cap \operatorname{ran} a^k$ and $\ker a^n \cap \operatorname{ran} a^{k+1}$ onto the same space. It follows that

$$\ker a^n \cap \operatorname{ran} a^k \subset \ker a^n \cap \operatorname{ran} a^{k+1} + \ker a^{n-1} \cap \operatorname{ran} a^k$$

from which point an inductive argument yields the desired result.

Proposition 6.6. *If $a \in L(V)$ is locally nilpotent and $\ker a \cap \operatorname{ran} a^k$ is finite–dimensional for some $k \in \mathbb{N}$, then a admits a canonical form.*

Proof. Suppose first $a \in L(V)$ is nilpotent. Using bases for succesive quotients of $\ker a$, $\ker a \cap \operatorname{ran} a$, $\ker a \cap \operatorname{ran} a^2$, $\ldots$, we can write $\ker a = \oplus_{i=0}^{n} K_i$ where $K_i \subset \operatorname{ran} a^i$ and $K_i \cap \operatorname{ran} a^{i+1} = \{0\}$ for $i \leq n$. For each i, we then find a subspace $M_i \subset V$ which a^i maps injectively onto K_i and set $V_i = \langle M_i \rangle$. By construction, the restriction of a to V_i is similar to a direct sum of $\dim K_i$ copies of the operator s_i from Example 6.2. The proof of this case is then completed by verifying that $V = \oplus V_i$.

In the general case, fix k so that

$$K_\omega \equiv \bigcap_{n \in \mathbb{N}} [\ \ker a \cap \operatorname{ran} a^n\ ] = \ker a \cap \operatorname{ran} a^k.$$

Applying Lemma 6.5, we see that $\ker a^n \cap \operatorname{ran} a^k \subset \operatorname{ran} a^{k+1}$ for each n, so a maps the range of a^k onto itself. Proceeding inductively, we can then construct subspaces $W_0 = K_\omega$, $W_1, W_2, \ldots$ such that for each i, the operator a maps W_{i+1} injectively onto W_i. Set $V_\omega = \cup_{n \in \mathbb{N}} W_i$. By construction, the restriction of a to V_ω is similar to a direct sum of finitely many copies of s_ω. One then uses a Zorn's lemma argument to find an invariant subspace M complementary to V_ω. Write b for the restriction of a to M, and observe that $\cap_{n \in \mathbb{N}} [\ker b \cap \operatorname{ran} b^n] = \{0\}$. Since one of the intersectands is finite–dimensional by hypothesis, it follows that $\ker b \cap \operatorname{ran} b^n = \{0\}$ for some n, and the proof is completed by applying the first case to b.

Corollary 6.7. *Every locally nilpotent operator which is not reflexive admits a canonical form.*

Proof. If a is locally nilpotent and non–reflexive, then Corollary 5.9 tells us that $\ker a \cap \operatorname{ran} a^k$ is one–dimensional for some $k \in \mathbb{N}$, and Proposition 6.6 applies.

The reader may object to the arbitrary nature of Definition 6.1. Thus while Corollary 6.7 is fairly definitive in saying that non–reflexive local nilpotents are well–behaved, Example 6.4 is scant evidence for the converse. The following discussion of Ulm invariants reveals Example 6.4 as "the tip of the iceberg" concerning intractability of reflexive local nilpotents.

Fix a locally nilpotent operator $a \in L(V)$. Following [12], we construct a family of subspaces of V indexed by ordinal numbers (starting at 0), by setting

$V_0 = V$,
$V_{\alpha+1} = aV_\alpha$, and
$V_\alpha = \cap_{\beta < \alpha} V_\beta$ when α is a limit ordinal.

For finite n we have $V_n = \operatorname{ran} a^n$. There must be a smallest ordinal λ for which $V_\lambda = V_{\lambda+1}$ and we avoid set theoretic difficulties by halting the construction at that point. Note that V_λ is in fact the largest subspace of V mapped onto itself by a.

Definition 6.8. The *Ulm invariants* for a locally nilpotent transformation a are given by

$$\mathcal{U}_\alpha(a) = \begin{cases} \dim \frac{\ker a \cap V_\alpha}{\ker a \cap V_{\alpha+1}}, & \text{for } \alpha < \lambda \\ \dim [\ \ker a \cap V_\lambda\ ], & \text{for } \ \alpha = \lambda. \end{cases}$$

The ordinal λ is referred to as the *length* of a.

For finite α, these similarity invariants agree with those of Display 6.1. When $\dim V \le \aleph_0$, our earlier block invariants can be recovered from the formula

$$\dim [\ \ker a \cap V_\beta\ ] \ = \ \sum_{\alpha \ge \beta} \mathcal{U}_\alpha(a), \qquad \beta \le \lambda.$$

Kaplansky proves that the Ulm invariants are complete if and only if the underlying space V has countable dimension. More to the point of the present discussion, even on F^ω, there are local nilpotents of each length $< \Omega$. Moreover, Ulm invariants of such operators can be arbitrarily specified subject to the mild restriction that there be infinitely many non–vanishing invariants between any two limit ordinals not exceeding λ (see Remark (d) on page 31 of [12]). From this perspective, we see that Example 6.4 is tame indeed with its length of $\omega + 1$ and its Ulm invariants being one for $\alpha \le \omega$.

We conclude the paper by observing that Ulm invariants lead to a simple characterization of those operators on F^ω which admit canonical forms; a variant of Example 6.4 shows that the dimensionality restriction cannot be dropped.

Theorem 6.9. *In order for a local nilpotent operator to admit a canonical form it is necessary that its length not exceed ω. If V has countable dimension, the condition is also sufficient.*

Proof. Necessity follows from Proposition 6.3(3) since $V_\omega = \cap_{n\in\mathbb{N}} \operatorname{ran} a^n$. For sufficiency, we have only to note that a has the same Ulm invariants as $\bigoplus_{0\le k\le\lambda} s_k^{(\mathcal{U}_k)}$.

Example 6.10. There is a locally nilpotent operator of length ω which does not admit a canonical form.

Construction. Take V to be the space of all lower–triangular infinite matrices which are supported on finitely many columns. Let a act on V by "shifting one column to the left", i.e.,

$$(av)_{i,j} = v_{i,j+1}, \quad v \in V \quad i, j \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, we have

$$\ker a \cap \operatorname{ran} a^k = \{\ v \in V : v \text{ is supported on the positions } (i, 1),\ i > k\}.$$

It follows that $V_\omega \equiv \cap_{k\in\mathbb{N}} \operatorname{ran} a^k = \{0\}$ and all the dimensions of Display 6.1 equal one. In view of Proposition 6.3, this means that if a had a canonical form, it would have to be $\oplus_{k\in\mathbb{N}} s_k$. This however is incompatible with the fact that $\dim V > \aleph_0$.

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