

## From Algebras of Normal Operators to Intersecting Hyperplanes

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Good theorems are often the basis for definitions. In [12], D. Sarason proved that every algebra of normal operators which is closed in the weak operator topology is reflexive. Analysis of Sarason's proof led the authors of [14], [9] and [7] to extend the notion of reflexivity to linear spaces of operators and to characterize "hereditarily reflexive" algebras of operators. The key property, called " $P_1$ " in [8], " $D_\sigma$ " in [7], " $(A_1)$ " in [3], and "elementary" in [1], has been expressed in terms of preduals and preannihilators. It is the purpose of the present paper to gain perspective by retreating to the more primitive concept of separation by linear functionals.

We begin with generalities on separation. A map  $g$  taking values in a topological space *separates* subsets  $A$  and  $B$  of its domain  $D$  if  $g(A)$  and  $g(B)$  have disjoint closures. A set  $G$  of such maps *separates*  $A$  and  $B$  if some  $g \in G$  does the separation. The  $G$ -hull of  $A$  denotes the set of all points in  $D$  which  $G$  fails to separate from  $A$ .

**PROPOSITION 1.** *Let  $S$  be a convex set in a topological vector space  $X$  and suppose  $G \subseteq X^*$ . Then the  $G$ -hull of  $S$  equals*

- (a)  $(S^\perp \cap G)^\perp$  if  $S$  is linear,
- (b) the closure of  $S$  if  $G = X^*$ .

**PROOF.** (b) is a version of the Hahn-Banach Theorem proved for example in [15, page 130]. For (a), note that  $g(S)$  is a linear manifold of scalars for each  $g \in G$ . Thus if  $g$  separates  $b$  from  $S$ , we must have  $S \subseteq \ker g$  while  $g(b) \neq 0$ .  $\square$

Let  $H$  be a Hilbert space, and write  $L$  for the algebra of bounded linear operators on  $H$ .  $A$  will denote an identity-containing subalgebra of  $L$ , not necessarily closed in any topology. For each pair of vectors  $x, y \in H$ , the

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vector functional  $x \otimes y$  in  $L^*$  is defined by setting  $\langle a, x \otimes y \rangle$  (the action of  $x \otimes y$  on  $a$ ) equal to the Hilbert space inner product of  $ax$  with  $y$ . We write  $F_k$  for the collection of functions in  $L^*$  which are sums of at most  $k$  vector functionals,  $F_\infty$  for the full linear span of  $F_1$ , and  $T$  for the norm closure of  $F_\infty$ . The  $F$  stands for finite rank and  $T$  stands for trace class, reflecting the fact that  $T$  can be identified with an ideal in  $L$  where rank has its traditional meaning; neither this nor the fact that  $T$  is the predual of  $L$  will play any role in the sequel.  $F_\infty$  and  $T$  induce topologies on  $L$  called the *weak operator topology* (w.o.t.) and the *ultraweak operator* or *weak\* topology* respectively.

**DEFINITION 2.** Let  $S$  be a convex set of operators.

- (a)  $S$  is *k-reflexive* if  $F_k$  separates each point of  $L \setminus S$  from  $S$ . ("reflexive" stands for "1-reflexive")
- (b)  $S$  is *elementary* if  $F_1$  and  $T$  are equally effective at separating points in  $S$  from convex subsets of  $S$ .

Our first job is to reconcile this with earlier definitions. Note that vectors in  $H$  can be thought of as maps from  $L$  into  $H$ . It follows directly from the definition that the  $H$ -hull of  $S$  is  $\{a \in L \mid ax \in \overline{Sx} \text{ for all } x \in H\}$ , often referred to as  $\text{ref } S$ .

**PROPOSITION 3.** Let  $S$  be a convex set of operators. Then the  $F_1$  and  $H$  hulls of  $S$  coincide.

**PROOF.** Apply Proposition 1(b) to  $H$  to conclude that  $H^*$  ( $\simeq H$ ) separates  $bx$  from  $Sx$  iff  $bx \notin \overline{Sx}$ .  $\square$

**COROLLARY 4.** Let  $A$  be an identity-containing algebra of Hilbert space operators.

- (a) The  $F_1$ -hull of  $A$  is  $\text{alg lat } A$ .
- (b) The  $F_\infty$ -hull of  $A$  is the w.o.t.-closure of  $A$ .
- (c) The  $T$ -hull of  $A$  is the weak\*-closure of  $A$ .
- (d) The  $L^*$ -hull of  $A$  is the norm closure of  $A$ .

**PROOF.** (a) For each  $x \in H$ , the (closed, linear) subspace  $\overline{Ax}$  is invariant under  $A$  and these cyclic subspaces generate  $\text{lat } A$  as a lattice. (b), (c), (d) are direct consequences of Proposition 1(b).  $\square$

In particular, the operator algebra  $A$  is reflexive iff  $A = \text{alg lat } A$ . Applying Proposition 1(a) we can also express  $\text{alg lat } A = (A^\perp \cap F_1)_\perp$ —this differs from the expression  $(A_\perp \cap F_1)^\perp$  of [1] only in that we have deliberately ignored the fact that  $T$  is the predual of  $L$ .

**PROPOSITION 5.** A linear subspace  $S$  of  $L$  is elementary iff  $T \subseteq S^\perp + F_1$ .

**PROOF.** Suppose  $S$  is elementary and let  $t \in T$ . If  $t = 0$ , we can write  $t = 0 + 0 \in S^\perp + F_1$ . Otherwise, choose  $a \in S$  with  $\langle a, t \rangle = 1$ . Since  $t$  separates  $a$  from the convex set  $C \equiv S \cap \ker t$ , there is an  $f \in F_1$  which also

does the separation. Thus  $C \subseteq \ker f$ ; multiplying  $f$  by a nonzero scalar if necessary, we can arrange  $\langle a, f \rangle = 1$ . Thus  $S = \text{span}(a, C) \subseteq \ker(t - f)$  so  $t \in S^\perp + F_1$ .

The converse is clear since  $t - f \in S^\perp$  means  $t$  and  $f$  agree on  $S$  and hence do the same job of separating points in  $S$  from convex subsets of  $S$ .  $\square$

The only novelty claimed for the foregoing discussion is the parallelism it highlights: reflexivity means good *external* separation by  $F_1$ , while elementarity means good *internal* separation by  $F_1$ . This symmetry is highlighted by the following notation.

**DEFINITION 6.** For  $G \subseteq J \subseteq L^*$ , write  $G \sim J$  in  $S$  if  $G$  and  $J$  are equally effective at separating points in  $S$  from convex subsets of  $S$ ;  $G \sim J$  outside  $S$  means  $G$  and  $J$  are equally effective at separating points outside of  $S$  from  $S$ , i.e., the  $G$ - and  $J$ -hulls of  $S$  coincide.

This perspective makes the “hereditary connection” clear.

**COROLLARY 7.** Suppose  $S$  is a reflexive set of operators. Then  $S$  is elementary iff every weak\* closed convex subset of  $S$  is reflexive.

**PROOF.** ( $\Rightarrow$ ) Let  $C$  be a weak\* closed convex subset of  $S$  and suppose  $b \in L \setminus C$ . By weak\* closure,  $T$  separates  $b$  from  $C$ . Since  $T \sim F_1$  both in and outside  $S$ , we see that  $F_1$  separates  $b$  from  $C$  regardless of whether  $b \in S$  or  $b \notin S$ .

( $\Leftarrow$ ) Suppose  $t \in T$  separates  $b \in S$  from  $C \subseteq S$ . The weak\* closure of  $C$  being reflexive, conclude that  $F_1$  separates  $b$  from  $C$ .  $\square$

The algebra  $\left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \in M_2(\mathbb{C}) \mid a, b \in \mathbb{C} \right\}$  is elementary, but not reflexive; in fact, it is not even known whether all elementary algebras have nontrivial invariant subspaces. Starting with S. Brown’s paper [4] however, elementarity (alias Property  $(A_1)$ ) has played a role in the proofs of important invariant subspace theorems. The reader is referred to C. Pearcy’s survey paper [10] and the references cited there for a discussion of these results. Here we concentrate on the most direct connection between elementarity and reflexivity, observed by D. Larson [8].

**PROPOSITION 8.** Let  $S$  be an elementary w.o.t.-closed subspace of operators. Then  $S$  is 3-reflexive.

**PROOF.** By hypothesis,  $F_1 \sim T$  in  $S$ , so surely  $F_1 \sim F_\infty$  in  $S$ . We will show  $F_3 \sim F_\infty$  outside  $S$ . Since  $F_\infty$  separates each point in  $L \setminus S$  from  $S$  by hypothesis, the same must then be true for  $F_3$  i.e.,  $S$  must be 3-reflexive.

To see that  $F_3 \sim F_\infty$  outside  $S$ , suppose  $F_3$  fails to separate  $b \in L$  from  $S$ . Define  $\phi: F_1 + S^\perp \rightarrow \mathbb{C}$  by  $\phi(f + t) = \langle b, f \rangle$ . Note first that the domain of  $\phi$  is the linear space  $F_\infty + S^\perp$  by Proposition 5. Moreover, if  $f_1 + t_1 = (f_2 + t_2) + (f_3 + t_3)$  with  $f_i \in F_1, t_i \in S^\perp$ , then  $(f_1 - f_2 - f_3) \in S^\perp \cap F_3$  whence  $\langle b, f_1 \rangle = \langle b, f_2 + f_3 \rangle$  since  $F_3$  does not separate  $b$  from  $S$ . This shows that  $\phi$  is well-defined and linear. Now  $t \in S^\perp$  implies  $\phi(t) = 0$  by definition, while

$t \in F_\infty$  implies  $\phi(t) = \langle b, t \rangle$  by linearity. This means  $b \perp S^\perp \cap F_\infty$ , so  $F_\infty$  fails to separate  $b$  from  $S$  too.  $\square$

This rearrangement of Larson’s original proof motivated the authors’ paper [2]; specifically, we wanted to examine the generality of the phenomenon “ $F_1 \sim F_\infty$  in  $S$  implies  $F_2 \sim F_\infty$  outside  $S$ ”. Few notational changes are needed for the general setting.  $S$  becomes a convex subset of a general Banach space  $L$ ; we take a general subset  $G$  of the dual space  $L^*$  to play the role of  $F_1$ . Since separation properties of linear functionals are not affected by scalar multiplication, we assume  $G$  is closed under scalar multiplication; we do not, of course, assume  $G$  is closed under addition. We write  $G_n$  for the set of functionals which can be expressed as sums of  $n$  or fewer members of  $G$ ,  $G_\infty$  for the full linear span of  $G$ , and  $T$  for the norm closure of  $G_\infty$  in  $L^*$ . No change is required in the notation and terminology of Definition 6.

Examination of the proofs of Propositions 5 and 8 above shows that  $G \sim G_\infty$  in  $S$  implies  $G_3 \sim G_\infty$  outside  $S$  for any linear  $S$ .

**THEOREM 9.** *If the (linear span of the ) convex set  $S$  is finite-dimensional and  $G \sim G_\infty$  in  $S$ , then  $G_3 \sim G_\infty$  outside  $S$ .*

This is the main result of [2]. The technical obstacle to the proof is the loss of Proposition 5: for general convex sets  $G \sim G_\infty$  in  $S$  does not imply  $G_\infty \subseteq S^\perp + G$ . In circumventing this difficulty, we need several concepts from convexity theory: supporting hyperplanes, directions of recession, imbedding in cones, relative interior, and exposed points. (We used [11] to learn about them.) In the sequel, we assume the field of scalars for  $L$  is  $\mathbf{R}$ ; the proof of Theorem 9 for complex  $L$  is easily reduced to this case.

**PROPOSITION 10.** *Suppose  $S$  is convex and  $G \subseteq J \subseteq L^*$  is closed under scalar multiplication. Then  $G \sim J$  in  $S$  iff given  $h \in J$ ,  $a \in S$  and  $\varepsilon > 0$ , there is a  $g \in G$  satisfying  $\langle z - a, h - g \rangle < \varepsilon$  for all  $z \in S$ .*

This is our substitute for Proposition 5, and the first step in the proof of Theorem 9; the characterization reduces to  $J \subseteq S^\perp + G$  when  $S$  is linear. Next comes imbedding: taking  $\hat{L} = L \oplus \mathbf{R}$ , setting

$$\hat{S} = \{(\lambda x, \lambda) | \lambda \geq 0, x \in S\} \cup \{(x, 0) | x \in L \text{ with } x + S \subseteq S\}$$

and taking  $\hat{G} = G \oplus \mathbf{R}$ , we reduce the proof of Theorem 9 to the conical case via the following proposition.

**PROPOSITION 11.** *For convex  $S \subseteq L$  and  $G \subseteq J \subseteq L^*$ , we have*

- (a)  $G \sim J$  in  $S$  iff  $\hat{G} \sim \hat{J}$  in the cone  $\hat{S}$ ,
- (b)  $G \sim J$  outside  $S$  iff  $\hat{G} \sim \hat{J}$  outside  $\hat{S}$ .

**OUTLINE OF THE PROOF OF THEOREM 9.** Suppose  $G \sim G_\infty$  inside the cone  $S$ , and  $G_3$  fails to separate  $b$  from  $S$ . To complete the proof, we need to show  $G_\infty$  fails to separate  $b$  from  $S$  too.

Set  $S^* = \{t \in L^* \mid t \leq 0 \text{ on } S\}$ . It follows from Proposition 10 that  $G_\infty \subseteq G + S^*$ . Define  $\phi_\pm: G_\infty \rightarrow \mathbb{C}$  by  $\phi_-(h) = \sup_{g \leq h \text{ on } S} \langle b, g \rangle$  and  $\phi_+(h) = \inf_{g \geq h \text{ on } S} \langle b, g \rangle$ . Then  $\phi_- \leq \phi_+$ . Of course  $\phi_+$  and  $\phi_-$  agree on  $G$ ; since  $G_3$  doesn't separate  $b$  from  $S$ , it follows that they are "monotone" in the sense that  $g_1, g_2, g_3 \in G$  with  $g_1 \leq g_2 + g_3$  on  $S$  implies  $\phi_\pm(g_1) \leq \phi_\pm(g_2) + \phi_\pm(g_3)$ .

Finite-dimensionality has not been invoked so far. It is now used (via existence of relative interior and exposed points) to prove that  $\phi_+ = \phi_- \equiv \phi$  on all of  $G_\infty$ . It is then easy to see that  $\phi$  remains monotone, hence linear on  $G_\infty$ . It follows that  $\phi(t) = \langle b, t \rangle$  for all  $t \in G_\infty$ . In particular, monotonicity gives  $\langle b, t \rangle \leq 0$  for all  $t \in S^* \cap G_\infty$  whence  $G_\infty$  cannot separate  $b$  from  $S$ .  $\square$

So far, we have concentrated on the relationship between  $G$  and  $G_\infty$ ; this corresponds to favoring the weak operator topology over the weak\* topology. Refinement of the open mapping techniques of [5], [6], and [13] can be used to rectify this.

**PROPOSITION 12.** *Suppose  $G$  is an analytic subset of  $L^*$  with  $G \sim T$  in the convex set  $S$ . Then  $G_3 \sim T$  outside  $S$  in the following cases.*

- (a)  $G$  dense in  $T$  (automatic if  $G$  linear),
- (b)  $S$  linear.

**COROLLARY 13.** *Let  $S$  be an elementary, weak\*-closed subspace of operators. Then  $S$  is 3-reflexive.*

**PROOF.**  $F_1 \sim T$  in  $S$  by hypothesis. Thus  $F_3 \sim T$  outside  $S$  by Proposition 12. Since  $T$  separates each point of  $L \setminus S$  from  $S$  by weak\*-closure, the same must be true for  $F_3$ .  $\square$

The following example shows that Theorem 9 is sharp in that  $G_3$  cannot be replaced by  $G_2$ . A fancier construction in [2] shows that " $G_2 \sim G_\infty$  outside  $S$ " may fail even if all the hypotheses of Theorem 9 and Proposition 12 are simultaneously satisfied.

**EXAMPLE 14.** Take  $S$  to be the plane  $z = 1$  in  $\mathbb{R}^3$ , and let  $G$  be the set of vectors  $(a, b, c) \in \mathbb{R}^3$  satisfying  $c = b - 3a$  if  $a, b$  have opposite signs while  $c = b - a$  otherwise. Actually  $G$  should be thought of as a subset of  $(\mathbb{R}^3)^*$ , but we make the usual identification between  $\mathbb{R}^3$  and its dual.

Now the action of any linear functional on  $S$  is completely determined by its first two coordinates; since such coordinates are unrestricted in  $G$ , we see that  $G \sim G_\infty$  in  $S$ . On the other hand,  $G_2$  intersects the  $z$ -axis only at the origin. Thus  $G_2$  fails to separate  $(0, 0, 0)$  from  $S$ , while  $(0, 0, 1) \in G_\infty$  does separate them, so we see that  $G_2 \not\sim G_\infty$  outside  $S$ .  $\square$

Example 14 has a geometric interpretation obtained by associating each  $(a, b, c) \in G$  with the line  $ax + by + c = 0$  in the plane. That  $G_2$  fails to separate  $(0, 0, 0)$  from  $S$  is reflected in the fact that any two members of this family  $G$  of lines intersect; that  $G_3$  does separate  $(0, 0, 0)$  from  $S$  corresponds to the fact that there are three members of  $G$  which are not concurrent.

Turning this idea around, Theorem 9 can be applied in a positive way to prove that certain families of hyperplanes are concurrent.

**COROLLARY 15.** *Let  $\mathcal{G}$  be a family of affine hyperplanes in  $\mathbf{R}^n$  whose normal vectors exhaust  $\mathbf{R}^n$ . If each three-member subset of  $\mathcal{G}$  is concurrent, then the whole family  $\mathcal{G}$  is concurrent.*

The hypothesis concerning normal vectors forces  $\mathcal{G}$  to be large; examples show that any weakening of it must be compensated for. Proposition 10 can be used to show that  $G \sim (\mathbf{R}^n)^*$  in the positive orthant  $(\mathbf{R}^n)^+$  iff  $G$  is dense in  $(\mathbf{R}^n)^*$ . This leads to the following variant of Corollary 15.

**COROLLARY 16.** *Let  $\mathcal{G}$  be a family of affine hyperplanes in  $\mathbf{R}^n$  whose normal vectors form a dense subset of  $\mathbf{R}^n$ . If every three-member subset of  $\mathcal{G}$  has a point of intersection in  $(\mathbf{R}^n)^+$ , then  $\mathcal{G}$  is concurrent.*

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