

# RESEARCH STATEMENT

BOBBE J. COOPER

## 1. INTRODUCTION

Cohomology is a powerful tool which allows us to study the maps between objects and to determine how simple objects fit together to form more complicated structures. My current research interests involve investigating support varieties which are defined using cohomological operations. Support varieties were first defined for modules over finite groups by Carlson in the early 1980's and have been a key to learning more about the representation theory of finite groups. The theory of the support varieties was later extended to the Frobenius kernel of an algebraic group scheme  $G$  (denoted  $G_1$ ) by Friedlander and Parshall [FP1]. For  $G_1$ , support varieties are intimately connected with the geometry of the restricted nullcone  $\mathcal{N}_1$  which can be identified with the spectrum of the cohomology of the restricted Lie algebra of  $G$ .

The tilting modules of  $G$  are rational  $G$ -modules for which both the module itself and its dual have good filtrations. These modules have many nice properties and are an integral part of understanding the representation theory of  $G$ . For example, knowing the characters of the indecomposable tilting modules is equivalent to the determination of the characters of the simple modules for  $G$ . The latter problem has a conjectural answer for large  $p$  through the Lusztig Conjecture, which uses the structure of Kazhdan-Lusztig polynomials for the affine Weyl group. There are also connections between the theory of tilting modules and the theory of Young modules for symmetric groups [D1, D2] when the underlying root system for  $G$  is of type  $A_n$ .

Humphreys' Conjecture (see Section 2.1 below) relates the support varieties of tilting modules to the combinatorics of Kazhdan-Lusztig two-sided cells of the affine Weyl group. Such connections between the Kazhdan-Lusztig combinatorics of the affine Weyl group and the representation theory of  $G$  are a rich and active area of research. My current research interests center on the calculation of support varieties for tilting modules, and its relation to the two-sided cells. This problem can be extended in many directions and relates to areas such as the representation theory and cohomology of algebraic groups, their Frobenius kernels, Lie algebras, and quantum groups, as well as combinatorics and algebraic geometry.

## 2. SUPPORT VARIETIES OF TILTING MODULES

**2.1. Algebraic Groups.** Let

$$R = \begin{cases} H^{2\bullet}(G_1, k) & \text{if char } k \neq 2 \\ H^\bullet(G_1, k) & \text{if char } k = 2. \end{cases}$$

where  $H^\bullet(G_1, k)$  is the cohomology ring of  $G_1$ . According to [FS],  $R$  is a commutative, finitely generated  $k$ -algebra. For a finite dimensional  $M \in G_1\text{-mod}$ , this allows us to define the *support variety*  $V_{G_1}(M)$  as follows. Yoneda composition defines an action of  $R$  on  $\text{Ext}_{G_1}^\bullet(M, M)$ . Let

$J = J_{G_1}(M)$  be the annihilator ideal in  $R$  for this action, and let  $V_{G_1}(M)$  be the maximum ideal spectrum of  $R/J$ .

In [Hum2, Section 12], J.E. Humphreys stated a conjecture for the support variety of an indecomposable tilting module when the highest weight of the module is regular (which implies that  $p \geq h$ ), for arbitrary reductive  $G$ . The basic idea is that the support variety only depends on the Kazhdan-Lusztig two-sided cell of the affine Weyl group  $W_p$  to which the weight belongs. Let  $O_w$  denote the nilpotent orbit corresponding to the two-sided cell of  $w \in W_p$  via the Lusztig bijection, and let  $C$  be the set of integral weights in the bottom alcove.

**Conjecture 2.1.1.** *Let  $\lambda \in X(T)_+$  be a regular dominant weight in  $w \cdot C$  for some  $w \in W_p$ . Then the support variety of the indecomposable tilting module of highest weight  $\lambda$  is the closure of  $O_w$ .*

The aforementioned conjecture requires that the weight be regular because it is not evident how to determine to which cell singular weights should belong.

My doctoral thesis work focused on computing support varieties of tilting modules over  $G_1$  when  $G$  is the special or general linear group over an algebraically closed field  $k$  of characteristic  $p > 0$ . In the process of my work, I formulated a new conjecture which predicts the support varieties of indecomposable tilting modules of highest weight  $\lambda$ , denoted  $T(\lambda)$ , for *all* primes  $p$  and *all* dominant weights  $\lambda$ , for type  $A_n$ . This new conjecture agrees with Humphreys' Conjecture for type  $A_n$  for regular weights when  $p \geq h$ . It also agrees with Donkin's Conjecture<sup>1</sup> for  $p = 2$  [D2].

Let  $\lambda \in X(T)_+$  be given; let  $\Phi^+$  denote the set of positive roots, and  $\Delta$  the set of simple roots. In order to state my conjecture, define

$$\Psi_\lambda = \{\alpha \in \Phi^+ : \langle \lambda + \rho, \alpha^\vee \rangle < p\}.$$

To each  $I \subseteq \Delta$ , we can assign a partition  $\pi(I)$  of  $n + 1$  as follows: We have

$$\Phi_I \cong A_{i_1-1} \times A_{i_2-1} \times \dots \times A_{i_t-1}$$

for some  $\{i_1, i_2, \dots, i_t\}, i_j \in \mathbb{Z}, i_j \geq i_{j+1} > 1$  for all  $j$ . Define  $\pi(I) = (i_1, i_2, \dots, i_t, 1, \dots, 1) \vdash n + 1$ .

Now, we define  $c(\lambda)$  as the minimal partition in the dominance order such that  $c(\lambda) \geq \pi(I)$  for all  $I$  satisfying  $\Phi_I^+ \subseteq \Psi_\lambda$ . It can be shown that such a minimal partition is unique.

We formulate a conjecture for the support variety of any indecomposable tilting module for  $G = GL_{n+1}$ . Note that  $\lambda$  may be either regular or singular, in contrast to Humphreys' Conjecture 2.1.1, and thus this conjecture also covers  $p < h$ . This conjecture agrees with Humphreys' Conjecture where they overlap (i.e.,  $\lambda$  regular,  $p \geq h$ , and  $G = GL_{n+1}$ ).

**Conjecture 2.1.2.** *Let  $\lambda \in X(T)_+$ . Then  $V_{G_1}(T(\lambda)) = \overline{\mathcal{O}_{c(\lambda)}}$ .*

**2.2. Main Results.** The reader is referred to [C], where portions of my Ph.D thesis have been written up in article form and submitted to *Advances in Mathematics*, for proofs of the results in this section.

My conjecture is especially amenable to the calculation of a lower bound for the support variety of a tilting module.

**Theorem 2.2.1.** *Let  $I \subseteq \Delta$  and  $\lambda \in X(T)_+$  with  $\Phi_I^+ \subseteq \Psi_\lambda$ . Then for all  $I$  such that  $\Phi_I^+ \subseteq \Psi_\lambda$ ,  $\overline{\mathcal{O}_{\pi(I)}} \subseteq V_{G_1}(T(\lambda))$ .*

<sup>1</sup>Note that the published version is not stated correctly.

In particular, if there exists an  $I$  such that  $\Phi_I^+ \subseteq \Psi_\lambda$  and  $c(\lambda) = \pi(I)$ , then this theorem shows that  $\overline{\mathcal{O}_{c(\lambda)}} \subseteq V_{G_1}(T(\lambda))$ . Thus one inclusion of the conjecture is proven in this common case. In general, it can be shown that this theorem reduces the problem of finding a lower bound of the support variety to showing that the variety is irreducible.

Given any partition  $\pi$  of  $n + 1$ , there exists at least one set of simple roots  $I \subseteq \Delta$  such that  $\pi(I) = \pi$ ; and given any such set of simple roots  $I$  with  $\pi(I) \leq (p^d, r)$  where  $n + 1 = pd + r$ , there exists a weight  $\lambda$  in  $X(T)_+$  such that  $\Psi_\lambda = \Phi_I^+$ . In fact, if  $I \neq \Delta$ , then there are infinitely many such  $\lambda$ . I can prove my conjecture for tilting modules whose highest weights fall in this infinite class, which intersects every cell.

**Theorem 2.2.2.** *If  $\lambda \in X(T)_+$  and  $\Psi_\lambda = \Phi_I^+$ , then  $V_{G_1}(T(\lambda)) = \overline{\mathcal{O}_{c(\lambda)}}$ .*

A partition is called a hook partition if it has at most one part bigger than 1. If  $\lambda$  is a weight for which  $c(\lambda)$  is a hook partition, the variety of  $T(\lambda)$  can be calculated using the formulation above. The following theorem provides an upper and lower bound for the support variety of *any* indecomposable tilting module. When  $c(\lambda)$  is a hook partition, the lower bound and upper bound agree and we have equality.

**Theorem 2.2.3.** *Let  $\lambda \in X(T)$  be such that  $c(\lambda) = (c_1, c_2, \dots, c_t)$  with  $c_t > 0$ . Let  $\sigma = (n + 1 - (t - 1), 1^{(t-1)})$ ; let  $\tau = (c_1, 1^{(n+1-c_1)})$ . Then  $\sigma$  is the least hook partition dominating  $c(\lambda)$  and  $\tau$  is the greatest hook partition dominated by  $c(\lambda)$ .*

- (a) *For any  $\lambda \in X(T)$ , we have  $\overline{\mathcal{O}_\tau} \subseteq V_{G_1}(T(\lambda)) \subseteq \overline{\mathcal{O}_\sigma}$ .*
- (b) *If  $c(\lambda)$  is a hook partition, then  $V_{G_1}(T(\lambda)) = \overline{\mathcal{O}_{c(\lambda)}}$ .*

We also prove that it suffices to look at weights in the restricted region  $X(T)_1 = \{\lambda \in X(T) : \langle \lambda + \rho, \alpha^\vee \rangle \leq p \text{ for all } \alpha \in \Delta\}$  in order to calculate the support varieties of all indecomposable tilting modules. This reduces the problem to a finite number of weights for any given  $n$ .

**Theorem 2.2.4.** *If  $V_{G_1}(T(\mu)) = \overline{\mathcal{O}_{c(\mu)}}$  for all  $\mu$  in the restricted region, then  $V_{G_1}(T(\lambda)) = \overline{\mathcal{O}_{c(\lambda)}}$  for all  $\lambda \in X(T)_+$ .*

**2.3.  $p=2$ .** Now we turn to small  $p$ . Note that if  $p < h$  then the interior of the alcoves contain no integral weights. As a result of this, certain cells no longer contain integral weights. For example, the bottom alcove is always a cell by itself and contains no integral weights whenever  $p < h$ . For smaller  $p$ , there will be fewer and fewer cells containing integral weights. In fact, the cells containing integral weights will be exactly the cells corresponding to partitions dominated by  $(p^d, r)$  where  $n + 1 = pd + r$ . Note that these partitions correspond exactly to the orbit closures which can appear as the support varieties of  $G_1$ -modules [CLNP].

This behavior limits the possible support varieties of tilting modules for small  $p$ , which actually simplifies the problem. In fact, I can prove my conjecture when  $p = 2$ , which proves Donkin's Conjecture.

**Theorem 2.3.1.** *When  $p = 2$ , we have  $V_{G_1}(T(\lambda)) = \overline{\mathcal{O}_{c(\lambda)}}$  for all  $\lambda \in X(T)_+$ .*

**2.4. Future Directions.** The methods in my Ph.D thesis will also be applicable to other small primes, and partitions with all parts less than 3. I have several preliminary results for these cases, and I will be looking to extend the theory in these directions.

In order to continue progressing on the problem of calculating support varieties of tilting modules for type  $A_n$ , I will need to use techniques involving translation functors (cf. [Jan2,

II 7]). Translation functors behave very nicely with respect to support varieties and tilting modules, so they are an ideal tool for studying this problem. I have general results about the effect of translation on tilting modules. In particular, I have a sufficient condition for weights in the closure of an alcove such that the support varieties of tilting modules of these highest weights are equal. Note that these functors are related to the theory of tensor ideals in the work of Ostrik and Bezrukavnikov [Ost2], [B2]. We can also use translation functors to obtain information about the character of a tilting module in certain specific cases.

A conjecture analogous to Conjecture 2.1.2 can be made for arbitrary root systems. I will investigate whether such a conjecture would be equivalent to Humphreys' Conjecture for regular weights, and also will look at computing support varieties of tilting modules for more arbitrary reductive groups. Preliminary investigation has shown that several of the new techniques I use for type  $A_n$  are also applicable in the general setting. It would be quite interesting, for example, if the  $p = 2$  result can be extended to arbitrary root systems.

The support variety gives us information about the cohomology of the tilting modules. With the results above, it may be possible to calculate the cohomology of specific tilting modules (cf. [Ost3] for one such example for quantum groups).

It is evident that there are connections between the structure of support varieties of tilting modules, and the characters of the simple modules for  $p > h$ . I plan to investigate extending this connection further, with an eye toward a Lusztig-style conjecture for algebraic groups when  $p$  is small.

### 3. QUANTUM GROUPS

**3.1.** Let  $\mathfrak{g}$  be a complex simple Lie algebra, and let  $U_\zeta$  be the quantum enveloping algebra associated to  $\mathfrak{g}$  for  $\zeta$  a complex primitive  $l$ th root of unity. The representation theory of  $U_\zeta$ , when  $l = p$ , is an approximation of the representation theory of the corresponding algebraic group  $G$  over characteristic  $p$ . We have  $u_\zeta$ , an analogue of the restricted enveloping algebra. The cohomology ring  $H^\bullet(u_\zeta, \mathbb{C})$  is a finitely generated algebra, which is shown for  $l > 3$  in [BNPP]. This allows us to define the support variety of a  $u_\zeta$  module by analogy with the algebraic group case. Likewise, one can define tilting modules for  $U_\zeta$  which are analogous to the tilting modules for  $G$ . Thus, one can examine Humphreys' Conjecture, or my new conjecture, in this setting.

One of the advantages of the quantum viewpoint is our ability to use tools from complex geometry. For example, in [ABG] there is a proof of the Lusztig Conjecture for the characters of the simple modules for  $l > h$  in this category. In fact, even for small  $l$  the characters of the simple modules have been calculated (cf. [Tan, Thm. 6.4, Thm. 7.1], [VV, p. 268]). Humphreys' Conjecture has been shown to be true for quantum groups by Ostrik [Ost1] (for type  $A_n$ ) and Bezrukavikov [B1] (in general), by using the validity of the Lusztig Conjecture for quantum groups for  $l > h$ .

**3.2. Future Directions.** I will investigate the plausibility of my conjecture in this setting, including looking at extending the results of my thesis to quantum groups. This will constitute an extension of the previous results in this setting in two separate directions – we will be investigating  $l < h$  for the first time, and applying more elementary and algebraic techniques to all the cases. Also, in the quantum group setting it is possible that we can do much

more than in the algebraic group setting, using the calculation of the characters of irreducible modules for quantum groups for  $l \leq h$  [Tan, Thm. 6.4, Thm. 7.1], [VV, p. 268].

#### 4. COMBINATORICS OF CELLS AND KAZHDAN-LUSZTIG POLYNOMIALS FOR AFFINE WEYL GROUPS

**4.1.** I have developed a new combinatorial algorithm for calculating the intersection of two-sided cells of the affine Weyl group of type  $A_n$  with the dominant chamber. It is a new generalization of the Robinson-Schensted correspondence to the affine Weyl group, which involves a variation of Young tableaux. Other attempts at this generalization can be found in [Shi] and [P]. My new method is an applicable algorithm which associates a finite tableau to each element  $w \in W_p$ ; the shape of the tableau is the partition corresponding via the Lusztig bijection to the cell containing  $w$ . Compare [P] which associates an infinite tableau to each element  $w$ , and [Shi], whose generalization is not easily applicable to any elements in  $W_p$  not in the finite Weyl group.

**4.2. Future Directions.** It would be of interest to extend this algorithm to the entire affine Weyl group of type  $A_n$ , and investigate other properties of the algorithm. In particular, is it possible to create a bijection between the affine Weyl group and some combinatorial structure which yields information about the cells? I plan to look for the existence of similar algorithms for arbitrary type, where the cells are not as well understood.

In a similar combinatorial vein, there is an algorithm for decomposition of tensor products of Weyl modules in [BK]. The calculations which I have already completed for the support varieties of tilting modules will yield information about the results of this algorithm, which have not yet been thoroughly studied. It is possible that such considerations would lead to information about an algorithm for the decomposition of tensor products of tilting modules.

#### REFERENCES

- [ABG] S. Arkhipov, R. Bezrukavnikov, V. Ginzburg, Quantum groups, the loop Grassmanian and the Springer resolution, *Journal of the AMS* **17** (2004), 595-678.
- [B1] R. Bezrukavnikov, Cohomology of tilting modules over quantum groups and t-structures on derived categories of coherent sheaves, *Invent. Math.* **166** (2006), 327-357.
- [B2] R. Bezrukavnikov, On tensor categories attached to cells in affine Weyl groups, preprint arXiv:Math.RT/0010089.
- [BNPP] C.P. Bendel, D.K. Nakano, B.J. Parshall, C. Pillen, Cohomology of quantum groups via the geometry of the nullcone, preprint (2007).
- [BK] J. Brundan, A. Kleshchev, Modular Littlewood-Richardson coefficients, *Mathematische Zeitschrift* **232** (1999), 287-320.
- [C] B. Cooper, On support varieties of tilting modules, submitted.
- [CLN] J.F. Carlson, Z. Lin, D.K. Nakano, Support varieties for finite Chevalley groups and classical Lie algebras, to appear in Trans. of AMS.
- [CLNP] J.F. Carlson, Z. Lin, D.K. Nakano, B.J. Parshall, The restricted nullcone, *Cont. Math.* **325** (2003), 51-75.
- [CM] D.H. Collingwood, W.M. McGovern, *Nilpotent Orbits in Semisimple Lie Algebras*, Van Nostrand Reinhold, 1993.
- [CPS1] E. T. Cline, B. J. Parshall, L. L. Scott, Finite dimensional algebras and highest weight categories, *J. Reine Angew. Math.* **391** (1988), 85-99.
- [CPS2] E. T. Cline, B. J. Parshall, L. L. Scott, Infinitesimal Kazhdan-Lusztig theories, *Cont. Math.* **139** (1992), 43-73.

- [CPS3] E. T. Cline, B. J. Parshall, L. L. Scott, Abstract Kazhdan-Lusztig theories, *Tohoku Math. J.* **45** (1993), 511-594.
- [D1] S. Donkin, On tilting modules for algebraic groups, *Mathematische Zeitschrift* **212** (1993), 39-60.
- [D2] S. Donkin, Tilting modules for algebraic groups and finite dimensional algebras, *A Handbook of Tilting Theory*, D. Happel, H. Krause (ed.), Cambridge University Press 2005.
- [FP1] E. Friedlander, B. Parshall, Support varieties for restricted Lie algebras, *Invent. Math.* **86** (1986), 553-562.
- [FP2] E. Friedlander, B. J. Parshall, Cohomology of Lie algebras and algebraic groups, *Amer. J. Math.* **108** (1986), 235-253.
- [FS] E.M. Friedlander, A.A. Suslin, Cohomology of finite group schemes over a field, *Invent. Math.* **127** (1997), no. 2, 209-270.
- [G] C. Greene, The structure of Sperner k-families, *Journal of Combinatorics Theory* **127** (1997), no. 2, 209-270.
- [Hum1] J.E. Humphreys, *Conjugacy Classes in Semisimple Algebraic Groups*, Math. Surveys and Monographs, AMS, vol. 43, 1995.
- [Hum2] J.E. Humphreys, Comparing modular representations of semisimple groups and their Lie algebras, *Modular Interfaces*, AMS/IP Stud. Adv. Math, **4** (1997), 69-80.
- [Jan] J.C. Jantzen, *Representations of Algebraic Groups*, Second Edition, American Mathematical Society, Providence R.I., 2003.
- [Jan2] J.C. Jantzen, Support varieties of Weyl modules, *Bulletin of the London Mathematical Society*, **19** (1987), 238-244.
- [K] H. Kraft, Parametrisierung von Konjugationsklassen in  $\mathfrak{sl}_n$ , *Math. Ann.*, **234** (1978), 209-220.
- [KL] D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.*, **53** (1979), 165-184.
- [NPV] D.K. Nakano, B.J. Parshall, D.C. Vella, Support varieties for algebraic groups, *J. Reine Angew. Math.* **547** (2002), 15-49.
- [Ost1] V. Ostrik, Support varieties for quantum groups, *Funct. Anal. Appl.* **32** (1998), 237-246.
- [Ost2] V. Ostrik, Tensor ideals in the category of tilting modules, *Transformation Groups*, **2**, No. 3 (1997), 279-287.
- [Ost3] V. Ostrik, Cohomology of subregular tilting modules for small quantum groups, *Compositio Math.*, **132**, No. 3 (2002), 283-287.
- [P] I. Pak, Periodic permutations and the Robinson-Schensted correspondence, preprint.
- [Par] B. Parshall, Cohomology of algebraic groups, *Proc. Symp. Pure Math.* **47** (1987), 233-248.
- [PS] B. Parshall, L. Scott, Koszul algebras and the Frobenius automorphism, *Quarterly J. Math Oxford* **46** (1995), 345-384.
- [PW] B. Parshall, J.P. Wang, Quantum Linear Groups, *Memoirs of the AMS* **89** no. 439 (1991).
- [SFB1] A.A. Suslin, E.M. Friedlander, C.P. Bendel, Infinitesimal one-parameter subgroups and cohomology, *Jour. AMS.* **10** (1997), 693-728.
- [Shi] J.Y. Shi, The Kazhdan-Lusztig Cells in Certain Affine Weyl Groups, *Lect. Notes in Math.* **1179**, Springer-Verlag, Berlin, Heidelberg, New York, 1986.
- [Tan] T. Tanisaki, Character formulas of Kazhdan-Lusztig type, *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, Fields Institute Communications **40** American Mathematical Society (2004), 261-276.
- [VV] M. Varagnolo, E. Vasserot, On the decomposition matrices of the quantized Shur algebras, *Duke Math. J.* **100** (1999), 267-297.