Singular Chern Classes of Schubert Cells and Varieties Via Small Resolution

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Outline

MACPHERSON AND MATHER CLASSES

- X – complex algebraic variety
- What is a singular Chern class? An assignment
  $X \mapsto c(X) \in H_* X$
  For smooth X we require: $c(X) = c(TX) \cap [X]$.
  - Let $\nu : \tilde{X} \to X$ be the Nash blowup
  - Chern-Mather class:

$$c_{\mathrm{M}}(X) = \nu_*(c(\tau) \cap [\tilde{X}])$$

  - Chern-Schwartz-MacPherson (CSM) class:

$$c_{\mathrm{SM}}(X)$$

GRASSMANNIANS

$$Gr(k, \mathbb{C}^n) = \mathrm{GL}_n(\mathbb{C})/P = \cup_\lambda X_\lambda^\circ$$

- Schubert cells X_λ° partition $Gr(k, \mathbb{C}^n)$
- Schubert varieties $X_\lambda := \overline{X_\lambda^\circ}$ are B orbit closures
- The X_λ are often singular
- $H_* Gr(k, \mathbb{C}^n)$ has a $\mathbb{Z}$-basis $\{[X_\lambda]\}_\lambda$ (Schubert basis)

- Question 1: Compute $c_{\mathrm{SM}}(X_\lambda)$ in the Schubert basis

CSM CLASSES OF CELLS

- $S \subset Gr(k, \mathbb{C}^n)$ locally closed, $S = X \backslash Y$ with $Y \subset X$ closed
- Set $c_{\mathrm{SM}}(S) := c_{\mathrm{SM}}(X) - c_{\mathrm{SM}}(Y)$

- Question 2: Compute $c_{\mathrm{SM}}(X_\lambda^\circ)$

$$c_{\mathrm{SM}}(X_\lambda) = \sum_{\mu \leq \lambda} c_{\mathrm{SM}}(X_\mu^\circ)$$

POSITIVITY CONJECTURE

Aluffi-Mihalcea

- Idea: Compute $c_{\mathrm{SM}}(X_\lambda^\circ)$ using the Bott-Samelson resolution
- Explicit formula for coefficients of $c_{\mathrm{SM}}(X_\lambda^\circ)$

- Conjecture: $c_{\mathrm{SM}}(X_\lambda^\circ)$ is a positive class
- Positive means: non-zero & non-negative coefficients in the Schubert basis

POSITIVITY RESULTS

Theorem (Aluffi-Mihalcea)

Let $X_\lambda \subset Gr(k, \mathbb{C}^n)$ with $k \leq 3$. Then, $c_{\mathrm{SM}}(X_\lambda^\circ)$ is positive.

Theorem (Aluffi-Mihalcea)

Let $X_\lambda \subset Gr(k, \mathbb{C}^n)$ and $X_{(b)} \subset \partial X_\lambda$. Then, The coefficient of $[X_{(b)}]$ in $c_{\mathrm{SM}}(X_\lambda^\circ)$ is non-negative.

SMALL RESOLUTIONS

Different Approach: Use the functorial behavior of c_{SM} and a "computable" resolution

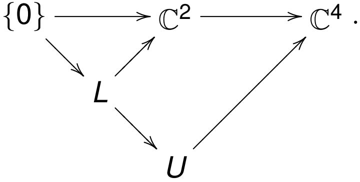
Theorem (Zelevinsky)

There exist small resolutions of singularities $\pi_\lambda : Z_\lambda \to X_\lambda$ for Schubert varieties in the Grassmannian.

EXAMPLE

$X_{(2,1)} \subset Gr(2, \mathbb{C}^4)$, the 3-dimensional singular Schubert variety

$$X_{(2,1)} = \{U \in Gr(2, \mathbb{C}^4) \mid \dim(U \cap \mathbb{C}^2) \geq 1\}$$



$$Z_{(2,1)} = \{(L, U) \in Gr(1, \mathbb{C}^4) \times Gr(2, \mathbb{C}^4) \mid L \subset U, L \subset \mathbb{C}^2\}$$

POSITIVITY RESULTS

Theorem (—)

- *$c_{\mathrm{SM}}(Z_\lambda)$ can be computed using tautological bundles on the ambient space of Z_λ*
- *There is an effective algorithm for computing $c_{\mathrm{SM}}(X_\lambda^\circ)$ in terms of:*
  1. $(\pi_\mu)_* c_{\mathrm{SM}}(Z_\mu)$ *for* $\mu \leq \lambda$
  2. *The Kazhdan-Lusztig polynomials $P_{\mu,\lambda}$*

Theorem (—)

Let $X_\mu \subset X_\lambda$ be codimension 1. Then, the coefficient of $[X_\mu]$ in $c_{\mathrm{SM}}(X_\lambda^\circ)$ is strictly positive.

COMBINATORIAL INTERPRETATION

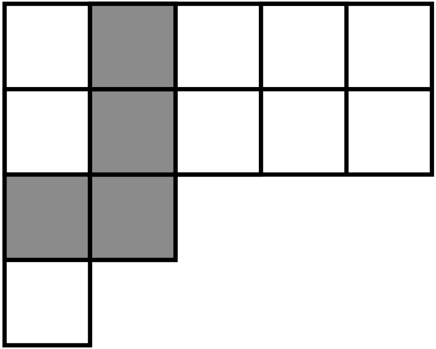


Figure: Codimension 1 coefficients count backwards hook length

MATHER CLASS AND NASH BLOWUP

- Characteristic cycle results of Bressler-Finkelberg-Lunts imply:

$$\mathrm{Eu}_\lambda(\mu) = P_{\mu,\lambda}(1)$$

- The local Euler obstruction for Schubert varieties is given by the stalks of intersection cohomology sheaves.

Corollary

$$(\pi_\lambda)_* c_{\mathrm{SM}}(Z_\lambda) = c_{\mathrm{M}}(X_\lambda)$$

Small resolutions compute the same class as the Nash Blowup.

NASH BLOWUP FOR SCHUBERT VARIETIES

Problem: Find a geometric proof of the corollary

- This would imply B-F-L result about characteristic cycles.
- Can the Nash blowup of a Schubert variety be realized as an “incidence variety” similar to the Zelevinsky resolutions?

Example:

$$\begin{array}{ccc} \widetilde{X}_{(2,1)} & \longrightarrow & Z^1_{(2,1)} \\ \downarrow & & \downarrow \pi_1 \\ Z^2_{(2,1)} & \xrightarrow{\pi_2} & X_{(2,1)} \end{array}$$

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