Computing Singular Chern Classes and Related Invariants for Schubert Varieties in a Grassmannian

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GOALS

- Compute singular Chern classes for certain complex algebraic varieties (main examples are Grassmannian Schubert Varieties)
- Chern-Schwartz-MacPherson (CSM) classes, Chern-Mather (CM) classes, and the local Euler obstruction.
- Apply to Grassmannian Schubert varieties (and cells)

The Chern-Mather class

- X – complex algebraic variety
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- Let $\nu : \tilde{X} \to X$ be the Nash blowup and $\underline{T}$ the Nash bundle.
- Chern-Mather (CM) class:

$$c_{\mathrm{M}}(X) = \nu_*(c(\underline{T}) \cap [\tilde{X}])$$

THE LOCAL EULER OBSTRUCTION

For $p \in X$, define

$$\mathrm{Eu}_X(p) := \int_{\nu^{-1}(p)} c(\underline{T}|_{\nu^{-1}(p)}) \cap S(\nu^{-1}(p), \tilde{X})$$

This is the <u>local Euler obstruction</u> at $p \in X$. It is an "integral character" of the singularity of X at p (if p is a sooth point, $\mathrm{Eu}_X(p) = 1$, in general it can be positive or negative).

THE CHERN-SCHWARTZ-MACPHERSON CLASS

- Chern-Schwartz-MacPherson (CSM) class:

$$c_{\mathrm{SM}}(X)$$

- $c_{\mathrm{SM}}(X)$ is a sum of CM classes of subvarieties of X, weighted by the local Euler obstruction.
- (Functoriality) CSM class commutes with the pushforward for constructible functions. $c_{\mathrm{SM}}(X) = c_*(1\!\!1_X)$ and if $f : Y \to X$ is proper the following diagram commutes:

$$\begin{array}{ccc} F_*Y & \xrightarrow{c_*} & H_*Y \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ F_*X & \xrightarrow{c_*} & H_*X. \end{array}$$

Let $M = \cup_i U_i$ be a smooth compact variety with fixed stratification by smooth, locally closed subsets U_i.

For each stratum U_i, one can define a class $c_{\mathrm{SM}}(U_i)$ such that

$$c_{\mathrm{SM}}(\overline{U_i}) = \sum_{U_j \subset \overline{U_i}} c_{\mathrm{SM}}(U_j).$$

By functoriality under proper maps, this is independent of how the characteristic function of U_i is decomposed.

Compute using resolution of singularities

Suppose $X = \overline{U_{i_0}}$ and $\pi : Z \to X$ is a resolution. Then by functoriality:

$$\pi_*(c_{\mathrm{SM}}(Z)) = \pi_* c_*(1\!\!1_Z) = c_* \pi_*(1\!\!1_Z)$$

The pushforward $\pi_*(1\!\!1_Z)$ is a sum over strata contained in X, weighted by the Euler characteristic of fibers of π. Thus,

$$\pi_*(c_{\mathrm{SM}}(Z)) = \sum_i f_i \cdot c_{\mathrm{SM}}(U_i) = c_{\mathrm{SM}}(U_{i_0}) + \sum_{U_i \subsetneq X} f_i \cdot c_{\mathrm{SM}}(U_i)$$

where $f_i = \chi(\pi^{-1}(p_i))$ for any $p_i \in U_i$.

We can solve for $c_{\mathrm{SM}}(U_{i_0})$ in the equation in terms of $\pi_* c_{\mathrm{SM}}(Z)$ and the classes of resolutions for $\overline{U_j} \subsetneq X$.

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$\Rightarrow$ We want to use this approach in situations where we understand the resolutions (and their fibers) explicitly.

$\Rightarrow$ For *certain* varieties X, this leads to calcultions of the CSM and CM classes (and the local Euler obstruction).

GEOMETRIC CONDITION I

- A resolution of singularities $\pi : Z \to X$ is small if for all $i > 0$,
$$\operatorname{codim}\{x \in X \mid \dim \pi^{-1}(x) \geq i\} > 2i.$$
- Then for $x \in X$,
$$\chi(\pi^{-1}(x)) = \chi_x(\mathrm{IC}(X))$$
where $\mathrm{IC}(X)$ denotes the intersection cohomology sheaf of X.

GEOMETRIC CONDITION II

$X \subset M$ is a subvariety of a smooth variety M with a fixed Whitney stratification $M = \cup_i U_i$.

- For $\mathcal{F}^\bullet$ a constructible sheaf on M, there is a cycle

$$\mathrm{CC}(\mathcal{F}^\bullet) = \sum_i m_i(\mathcal{F}^\bullet)\left[\overline{T^*_{U_i} M}\right]$$

  called the characteristic cycle.
- $m_i(\mathcal{F}^\bullet)$ is called the microlocal multiplicity of $\mathcal{F}^\bullet$ along U_i.
- Let $X = \overline{U_{i_0}}$. We say X has *irreducible characteristic cycle* if $\mathrm{CC}(\mathrm{IC}(X)) = \left[\overline{T^*_{U_{i_0}} M}\right]$.

THEOREM (–)

Let X be a complex algebraic variety which admits a small resolution $\pi : Z \to X$ and whose characteristic cycle is irreducible. Then, the Chern-Mather class of X equals the push-forward of the Chern class of any small resolution $\pi : Z \to X$:

$$c_{\mathrm{M}}(X) = \pi_*(c(TZ) \cap [Z]).$$

The proof involves using the Microlocal Index Theorem to identify local Euler obstructions with invariants coming from intersection cohomology...

SCHUBERT VARIETIES

Consider a Schubert Variety in a Grassmannian:

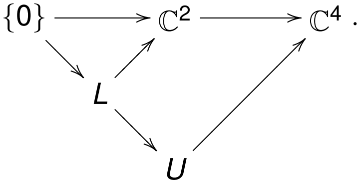
$$X_\lambda \subset Gr(k, \mathbb{C}^n).$$

- Goal: compute $c_{\mathrm{SM}}(X_\lambda)$ and $c_{\mathrm{M}}(X_\lambda)$ in the Schubert basis of $H_* Gr(k, \mathbb{C}^n)$
- (Explicit) small resolutions have been constructed by Zelevinsky
- Bressler-Finkelberg-Lunts have proved that X_λ has an irreducible characteristic cycle

Example

$X_{(2,1)} \subset Gr(2, \mathbb{C}^4)$, the 3-dimensional singular Schubert variety

$$X_{(2,1)} = \{U \in Gr(2, \mathbb{C}^4) \mid \dim(U \cap \mathbb{C}^2) \geq 1\}$$



Small resolution:

$$Z_{(2,1)} = \{(L, U) \in Gr(1, \mathbb{C}^4) \times Gr(2, \mathbb{C}^4) \mid L \subset U, L \subset \mathbb{C}^2\}$$

The pushforward of $1\!\!1_{Z_{(2,1)}}$:

$$\pi_*(1\!\!1_{Z_{(2,1)}}) = 1\!\!1_{X_{(2,1)}} + 1\!\!1_{X_{(0)}},$$

thus

$$\begin{aligned} c_{\mathrm{SM}}(X_{(2,1)}) &= \pi_*(c_{\mathrm{SM}}(Z_{(2,1)})) - c_{\mathrm{SM}}(X_{(0)}) \\ &= \pi_*(c_{\mathrm{SM}}(Z_{(2,1)})) - [\text{pt.}] \end{aligned}$$

By the proposition,

$$c_{\mathrm{M}}(X_{(2,1)}) = \pi_*(c_{\mathrm{SM}}(Z_{(2,1)})).$$

In general, the resolutions Z_λ are subvarieties of a product of Grassmannians (and are themselves iterated Grassmannian bundles over Grassmannians)

$$Z_\lambda \subset \prod_i Gr(k_i, \mathbb{C}^n).$$

- We express TZ_λ and the pushforward of $c(TZ_\lambda) \cap [Z_\lambda]$ in terms of universal bundles on the $Gr(k_i, \mathbb{C}^n)$.
- This gives an algorithm for computing $(\pi_\lambda)_*(c_{\mathrm{SM}}(Z_\lambda))$ explicitly.

Table 1. CSM Classes of Schubert Cells Contained in $Gr(3, \mathbb{C}^6)$

	α_0	α_1	α_2	α_3	β_0	α_4	α_5	β_1	α_6	α_7	β_2	β_3	α_8	γ_0	β_4	α_9	γ_1	β_5	γ_2	γ_3
α_0	1	5	12	12	20	20	34	54	54	31	66	57	57	27	75	27	27	27	9	1
α_1	.	1	4	4	8	8	15	27	27	17	39	34	34	18	51	18	21	21	8	1
α_2	.	.	1	.	.	3	4	8	11	7	19	15	18	9	31	12	15	16	7	1
α_3	.	.	.	1	3	.	4	11	8	7	19	18	15	12	31	9	16	15	7	1
β_0	.	.	.	.	1	.	.	4	.	.	8	7	.	8	15	.	12	9	6	1
α_4	.	.	.	.	.	1	.	.	4	.	8	.	7	.	15	8	9	12	6	1
α_5	.	.	.	.	.	.	1	3	3	3	8	8	8	6	18	6	11	11	6	1
β_1	.	.	.	.	.	.	.	1	.	.	3	3	.	4	8	.	8	6	5	1
α_6	.	.	.	.	.	.	.	.	1	.	3	.	3	.	8	4	6	8	5	1
α_7	.	.	.	.	.	.	.	.	.	1	.	3	3	3	8	3	7	7	5	1
β_2	.	.	.	.	.	.	.	.	.	.	1	.	.	.	3	.	4	4	4	1
β_3	.	.	.	.	.	.	.	.	.	.	.	1	.	2	3	.	5	3	4	1
α_8	.	.	.	.	.	.	.	.	.	.	.	.	1	.	3	2	3	5	4	1
γ_0	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	3	.	3	1
β_4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	2	2	3	1
α_9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	3	3	1
γ_1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	2	1
β_5	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	2	1
γ_2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1
γ_3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

Positivity

Given λ, let X_λ° denote the Schubert Cell.

Theorem (Aluffi-Mihalcea, Mihalcea)

Let $X_\lambda \subset Gr(k, \mathbb{C}^n)$ with $k \leq 3$. Then, $c_{\mathrm{SM}}(X_\lambda^\circ)$ is positive.

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Conjecture ([1]): For arbitrary $X_\lambda \subset Gr(k, \mathbb{C}^n)$, the class $c_{\mathrm{SM}}(X_\lambda^\circ)$ is positive.

$\Rightarrow$ This implies the weaker statement that all $c_{\mathrm{SM}}(X_\lambda)$ are positive.

Positivity Results: Special Cases

1. (Easy) Suppose $\lambda = (a^b)$ for $a, b > 0$. Then X_λ is a Grassmannian itself and $c_{\mathrm{SM}}(X_\lambda)$ and $c_{\mathrm{SM}}(X_\lambda^\circ)$ are positive.
2. (Slight Generalization) Suppose $\lambda = (a + p, a^b)$ for $a, b, p > 0$. Then, X_λ has singular locus a Grassmannian and $c_{\mathrm{SM}}(X_\lambda)$ and $c_{\mathrm{SM}}(X_\lambda^\circ)$ are positive.
3. (Different Direction) Suppose λ is arbitrary and X_μ has small (≤ 2) codimension in X_λ. Then the coefficient of $[X_\mu]$ in $c_{\mathrm{SM}}(X_\lambda^\circ)$ is strictly positive.

Combinatorial Interpretation

The coefficient of $[X_{(5,5,1,1)}]$ in the CSM class of $X^{\circ}_{(5,5,2,1)}$ is 4:

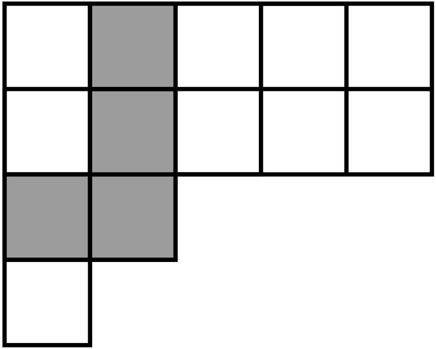

Figure: Codimension 1 coefficients count anti-hook length

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