Normality of Enhanced Nilpotent Orbit Closures

Benjamin F Jones[1]

[1]Department of Mathematics
University of Georgia

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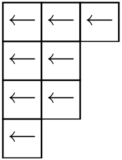
Nilpotent Matrices

- $G = \mathrm{GL}_n(\mathbb{C})$
- $\mathcal{N} = \{x \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid x^k = 0 \quad \text{for some} \quad 0 \leq k \leq n\}$
- $\mathcal{N}$ is stable under the conjugation action of G and there are finitely many orbits.
- Parametrization of the orbits: The orbits (nilpotent orbits) are in bijection with partitions λ of n
- Orbit $\mathcal{O}_\lambda$ corresponds to $G.x$ where x is nilpotent with Jordan blocks of size $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)}$.

Nilpotent Matrices

- Diagram: $\lambda = (3, 2, 2, 1)$



- Action of $x \in \mathcal{O}_\lambda$ moves basis vectors to the left, annihilates basis vectors in the left-most column

Representation Theory and $\mathcal{N}$

- Geometry of $\overline{\mathcal{O}_\lambda}$ is important in (geometric) representation theory:
  1. Springer Correspondence: representation theory of the Weyl groups
  2. K-theory of $\mathcal{N}$: representation theory of Hecke algebras
  3. Cohomology of quantum groups
  4. Unipotent character sheaves
  5. etc..
- The varieties $\overline{\mathcal{O}_\lambda}$ are well studied and much is known about their geometry.

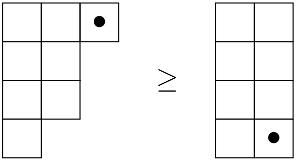
Geometry of $\overline{\mathcal{O}_\lambda}$

Closure inclusion relations:

- (Hesselink) $\mathcal{O}_\mu \subset \overline{\mathcal{O}_\lambda}$ iff. $\mu \leq \lambda$, i.e.

$$\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i$$

  for all i.
- Diagram:



Intersection Cohomology

Resolution of singularities exist: There exists a subset of simple roots $J \subset \Delta$ such that $\mathcal{O}_\lambda = G.\mathfrak{u}_J$ and

$$G \times_{P_J} \mathfrak{u}_J \xrightarrow{\pi} \mathcal{N},$$

where π is induced by $(g, x) \mapsto g.x$, is a *semi-small* resolution of singularities of $\mathcal{O}_\lambda$.

Theorem (Lusztig)

For $x \in \mathcal{O}_\mu$,

$$P_{\lambda,\mu}(t) := t^d \sum_i \dim \mathcal{H}^i_x(\mathrm{IC}(\overline{\mathcal{O}_\lambda}))t^i$$

is the Kostka polynomial corresponding to the pair λ, μ.

Normality

In type A, nilpotent orbit closures are *normal*, i.e. their cordinate rings are integrally closed.

Theorem (Kraft-Procesi)

There exists an irreducible variety Z_λ and a surjective map $f : Z_\lambda \to \overline{\mathcal{O}_\lambda}$ such that

1. *f is equivariant for a $H \times G$ action, where we let $H = \prod_i \mathrm{GL}_{n_i}(\mathbb{C})$ act on $\overline{\mathcal{O}_\lambda}$ trivially,*
2. *f is an invariant theory quotient for H,*
3. *Z_λ is a complete intersection and regular in codimension 1, hence it is normal.*

Normality

- Normality of Z_λ and the properties of f imply that $\overline{\mathcal{O}_\lambda}$ is normal.
- $\overline{\mathcal{O}_\lambda}$ normal implies that for $x \in \mathcal{O}_\lambda$:

$$\mathbb{C}[\overline{\mathcal{O}_\lambda}] \cong \mathbb{C}[\mathcal{O}_\lambda] \cong \mathbb{C}[G]^{G_x}.$$

Normality

The construction of Z_λ is related to what are now known as (type-A) quiver varieties.

Other results:

- (Kostant) Original proof that $\mathcal{N}$ (closure of the regular orbit) is normal
- (Hesselink, Vinberg-Popov) Small nilpotent orbits are normal
- (KP) Determination of normality for the classical Lie algebras
- (Kraft, Broer, Sommers) Determination of normality in types G_2, F_4, and E_6 and partial results in E_7, E_8.

Enhanced Nilpotent Orbits

Definition

The “Enhanced Nilpotent Cone” is $V \times \mathcal{N}$ where $V = \mathbb{C}^n$.

- (Bernstein) The set of orbits of G on $V \times \mathcal{N}$ is finite.
- Studying G-orbits in $V \times \mathcal{N}$ involves interesting geometry and combinatorics.
- The geometry is similar to type A nilpotent orbit closures.
- The combinatorics are similar to type C nilpotent orbit closures.

Motivation

The geometry and combinatorics of orbit closures in $V \times \mathcal{N}$ are closely related to orbit closures in Syu Kato's "Exotic Nilpotent Cone", $\mathfrak{N}$, and also appear in the theory of mirabolic character sheaves.

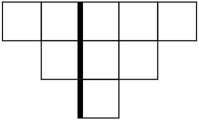
- The geometry of orbit closures in $\mathfrak{N}$ is the main ingredient in Kato's "Exotic" Springer correspondence.
- Equivariant K-theory of $\mathfrak{N}$ is the main tool in Kato's "Exotic" Delign-Langlands correspondence.
- (Finkelberg-Ginzburg-Travkin) Enhanced nilpotent orbit closures themselves are involved in classifying unipotent character sheaves on $G \times V$.

Properties of Enhanced Nilpotent Orbits

- Orbits are parametrized as follows: let $(v, x) \in V \times \mathcal{N}$ with $x \in \mathcal{O}_\lambda$. Then, there is a basis $\{v_{i,j}\}$ of V such that:
  1. $\{v_{i,j}\}$ is a Jordan basis for x.
  2. $v = \sum_{i=1}^{\ell(\mu)} v_{i,\mu_i}$ where μ is a partition such that $\nu := \lambda - \mu$ defines a partition.
- The pair $(\mu; \nu)$ is called a *bipartition*.
- (Achar-Henderson, Travkin) The set of orbits of G in $V \times \mathcal{N}$ is in bijection with the set of biparitions $(\mu; \nu)$ such that $|\mu| + |\nu| = n$.

Properties of Enhanced Nilpotent Orbits

Example: The (bi)partition diagram for $(2,1);(3,2,1)$



Properties of Enhanced Nilpotent Orbits

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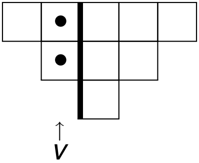

Here, $x^2.v = 0$.

Properties of Enhanced Nilpotent Orbits

The closures of enhanced nilpotent orbits have a simple description:

Proposition (Travkin)

$(v, x) \in \overline{\mathcal{O}_{\mu;\nu}}$ iff. there exists a $|\mu|$-dimensional subspace $W \subset V$ such that:

1. *$v \in W$,*
2. *W is invariant under x: $x(W) \subset W$,*
3. *$x|_W$ has Jordan type dominated by μ,*
4. *and the endomorphism induced by x on V/W has Jordan type dominated by ν.*

Properties of Enhanced Nilpotent Orbits

Achar and Henderson have described the orbit closure inclusion relations, constructed (semi-small) resoslutions, and computed the local intersection cohomology. Also...

- The enhanced nilpotent cone embeds in Kato's "Exotic Nilpotent Cone", $\mathfrak{N}$.
- The embedding is equivariant and orbits in $V \times \mathcal{N}$ correspond bijectively with orbits in $\mathfrak{N}$.
- The main motivation for studying enhanced orbit closures is that they are simpler, for example they admit natural resolutions of singularities, and they (conjecturally) capture the essential geometry of orbit closures in the Exotic Cone.

Intersection Cohomology

$\Rightarrow$ (Achar-Henderson) Poincare polynomials of the IC sheaves of orbit closures coincide with generalized Kostka polynomials introduced by Shoji.

$\Rightarrow$ Conjecture: The same polynomials with degrees doubled are the Poincare polynomials of exotic nilpotent orbit closures.

Normality of Enhanced Nilpotent Orbits

(joint work with Achar and Henderson)

Proposition (Achar-Henderson-J)

Let $(\mu; \nu)$ be a biparition of n such that $\ell(\mu)$ is less than or equal to the multiplicity of ν_1 (i.e. all the columns of ν are at least as long as the longest column of μ). Then, the orbit closure $\overline{\mathcal{O}_{\mu;\nu}}$ is normal.

The proof identifies $\overline{\mathcal{O}_{\mu;\nu}}$ as a GIT quotient of an "enhanced" type A quiver variety.

Conjecture: All enhanced nilpotent orbit closures are normal.

Normality of Enhanced Nilpotent Orbits

- The hypothesis in the proposition covers 13 out of 20 enhanced nilpotent orbit closures in $\mathbb{C}^4 \times \mathfrak{sl}_4(\mathbb{C})$.
- Normality also holds for orbit closures with $\nu = \emptyset$ and whenever $\lambda = \mu + \nu$ has only 2 columns. Along with the orbits covered by the proposition, we know that 17 out of 20 orbit closures in $\mathbb{C}^4 \times \mathfrak{sl}_4(\mathbb{C})$ are normal.
- We know that normality doesn't hold in general for orbit closures in $\mathfrak{N}$ (for example the ordinary nilpotent orbit closures in type C are contained in it).

Enhanced Type A Quiver Varieties

Example: When $r_1 \leq r_2$,

$$\overline{\mathcal{O}_{(1^{r_1});(1^{r_2})}} = \{(v, x) \in V \times \mathcal{N} \mid x^2 = 0,\ xv = 0,\ \dim(\mathrm{im}(x) + \mathbb{C}v) \leq r_1\}$$

is normal, being isomorphic to the GIT quotient of

$$\Lambda_{(r_1,r_2),0,1} = \{u \in \mathbb{C}^{r_1} \underset{B_1}{\overset{A_1}{\rightleftarrows}} \mathbb{C}^{r_1+r_2} \mid B_1 A_1 = 0\},$$

with the quotient map sending (u, A_1, B_1) to $(A_1(u), A_1 B_1)$.

References

- P. N. Achar and A. Henderson, *Orbit closures in the enhanced nilpotent cone*, Adv. Math. **219** (2008), no. 1, 27–62.
- H. Kraft and C. Procesi, *Closures of conjugacy classes of matrices are normal*, Invent. Math. **53** (1979), 224–247.
- A. Maffei, *Quiver varieties of type A*, Comment. Math. Helv. **80** (2005), 1–27.