UNIVERSITY OF GEORGIA

DEPARTMENT OF MATHEMATICS

2200 CALCULUS 1

Fall 2005, Kazancı

FINAL 12/14/05

Name : SOLUTIONS

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
6		25
7		10
Total		85

This is a 90 minute test. No books, notes or calculators are permitted. You may use the back of the page for additional space. Please show all your work on all problems. Good Luck!

1. (10 pts) Find the indicated limit if it exists. Give reasons why the limit does or does not exist. Find the right limit and the left limit if necessary.

(a) (5 pts)

$$\lim_{x\to 4} \sqrt[3]{3\sqrt{x^3}+20\sqrt{x}}$$

$$\lim_{x\to 4} \sqrt[3]{3\sqrt{x^3}+20\sqrt{x}} = \sqrt[3]{3\sqrt{64}+20\sqrt{4}}$$

$$= \sqrt[3]{3\cdot 8+20\cdot 2}$$

$$= \sqrt[3]{64} = \boxed{4}$$

(b) (5 pts)

$$\lim_{x\to 0} \frac{(3+x)^{-1}-(3-x)^{-1}}{x}$$

$$\lim_{x\to 0} \frac{\frac{1}{3+x}-\frac{1}{3-x}}{x}$$

$$= \lim_{x\to 0} \frac{(3-x)-(3+x)}{x(3-x)(3+x)}$$

$$= \lim_{x\to 0} \frac{\cancel{3}-x-\cancel{3}-x}{x(3-x)(3+x)}$$

$$= \lim_{x\to 0} \frac{-2\cancel{x}}{\cancel{x}(3-x)(3+x)}$$

$$= \lim_{x\to 0} \frac{-2}{(3-x)(3+x)} = \cancel{\lim} = \boxed{\frac{-2}{9}}$$

2. (10 pts) Find the derivative of the given functions.

(a) (5 pts)

$$f(x) = \frac{1}{2}\tan^2 x + \sin\sqrt{1+x^3}$$

$$f'(x) = \frac{1}{\cancel{2}}\cancel{2}\tan x \sec^2 x + \cos\sqrt{1+x^3}\ \frac{1}{2}(1+x^3)^{-\frac{1}{2}}(3x^2)$$

$$\boxed{f'(x) = \tan x \sec^2 x + \frac{3x^2\cos\sqrt{1+x^3}}{2\sqrt{1+x^3}}}$$

(b) (5 pts)

$$f(x) = 3^{log_2(1-x)}$$

$$f'(x) = 3^{\log_2(1-x)}\ln 3\ \frac{1}{(1-x)\ln 2}\cdot(-1)$$

$$\boxed{f'(x) = -\frac{3^{\log_2(1-x)}\ln 3}{(1-x)\ln 2}}$$

3. (10 pts) Apply the second derivative test to find the local maxima and local minima of the following function.

$$f(x) = x^3 - 3x^2$$

$f'(x) = 3x^2 - 6x = 3x(x-2) = 0$

$x = 0$, $x = 2$

are critical values

$f''(x) = 6x - 6$

$f''(0) = -6 < 0 \leftarrow$ local max. at $x = 0$

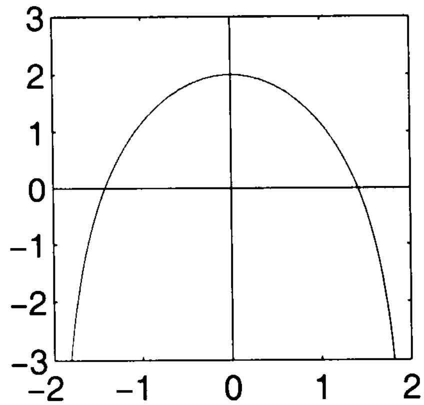
$f''(2) = 6 > 0 \leftarrow$ local min. at $x = 2$.

4. (10 pts.) Three functions and their derivatives are given below. Identify them by filling out the spaces below with numbers between 1 and 6.

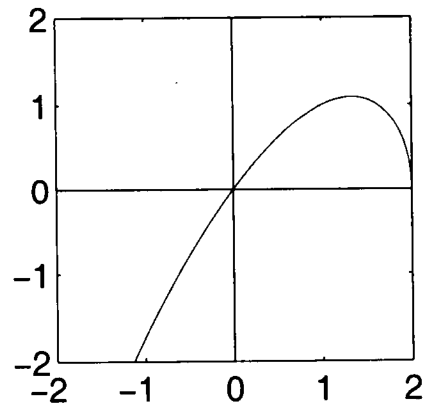
$f(x) =$ 2 $f'(x) =$ 5 $g(x) =$ 4 $g'(x) =$ 1 $h(x) =$ 6 $h'(x) =$ 3

(order is not important)

(1)



(2)



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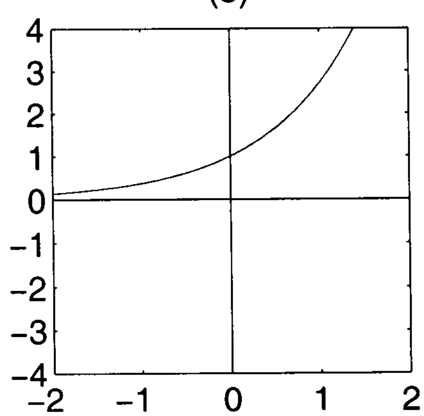

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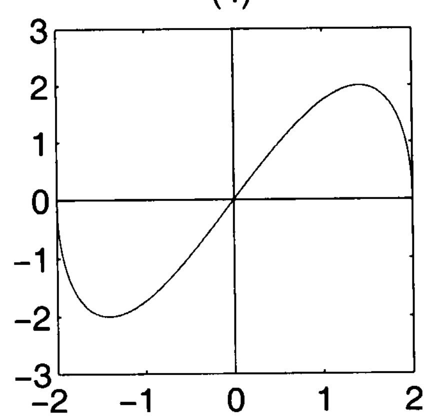
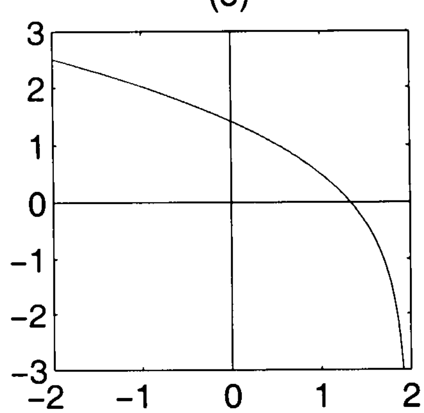
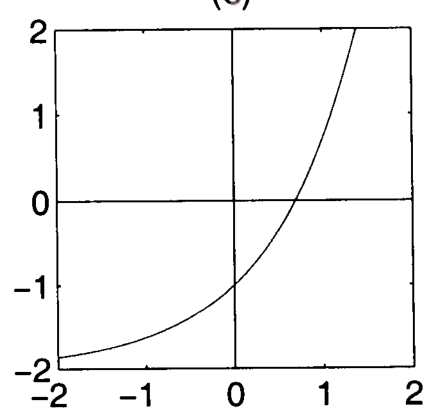


(5)



(6)



5. (10 pts.) Show that $f(x) = (x-1)^{2/3}$ satisfies the the hypothesis of the mean value theorem on the interval $[1, 2]$, and find all numbers c in that interval that satisfy the conclusion of that theorem.

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

$$\frac{2}{3}(c-1)^{-\frac{1}{3}} = \frac{(2-1)^{2/3} - (1-1)^{2/3}}{2-1} = \frac{1^{2/3} - 0}{1} = 1$$

$$(c-1)^{-\frac{1}{3}} = \frac{3}{2}$$

$$(c-1)^{\frac{1}{3}} = \frac{2}{3}$$

$$c - 1 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$c = 1 + \frac{8}{27} = \boxed{\frac{35}{27}}$$

6. (25 pts)

$$f(x) = \frac{x-1}{x^2}$$

(a) (3 pts) Find the x and y-intercepts.

x-intercepts :

$f(x) = \frac{x-1}{x^2} = 0 \implies x - 1 = 0 \implies \underline{x = 1}$

$f(1) = \frac{1-1}{1} = 0$

$\boxed{(1, 0) \text{ is an } x\text{-intercept}}$

y-intercepts :

$f(0) = \frac{-1}{0}$ DNE $\boxed{\text{No } y\text{-intercepts}}$

(b) (4 pts) Search for vertical and horizontal asymptotes.

H.A. : $y = 0$

V.A. : $x = 0$

(c) (6 pts) Find intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing. Search for local extrema.

$$f'(x) = \frac{1 \cdot x^2 - (x-1)\,2x}{x^4} = \frac{x^2 - 2x^2 + 2x}{x^4}$$

$$= \frac{-x^2 + 2x}{x^4} = \frac{x(2-x)}{x^4} = \frac{2-x}{x^3}$$

$$f'(x) = \frac{2-x}{x^3}$$

$f'(x)$: $-$ 0 $+$ 2 $-$

$f(x)$: $\searrow$ $\nearrow$ $\searrow$

$f(0)$ DNE

$x=0$ V.A.

$f(2) = \frac{2-1}{2^2} = \boxed{\frac{1}{4}}$ ← local max

$\boxed{\text{local max at } (2, 1/4)}$

(d) (6 pts) Find the intervals on which $f(x)$ is concave-up and the intervals on which $f(x)$ is concave-down. Search for inflection points.

$$f''(x) = \frac{-1 \cdot x^3 - (2-x)\,3x^2}{x^6} = \frac{-x^3 - 6x^2 + 3x^3}{x^6}$$

$$= \frac{2x^3 - 6x^2}{x^6} = \frac{2x^2(x-3)}{x^6} = \frac{2(x-3)}{x^4}$$

$$f''(x) = \frac{2(x-3)}{x^4}$$

$f''(x)$: $-$ 0 $-$ 3 $+$

$f(x)$: $\frown$ $\frown$ $\smile$

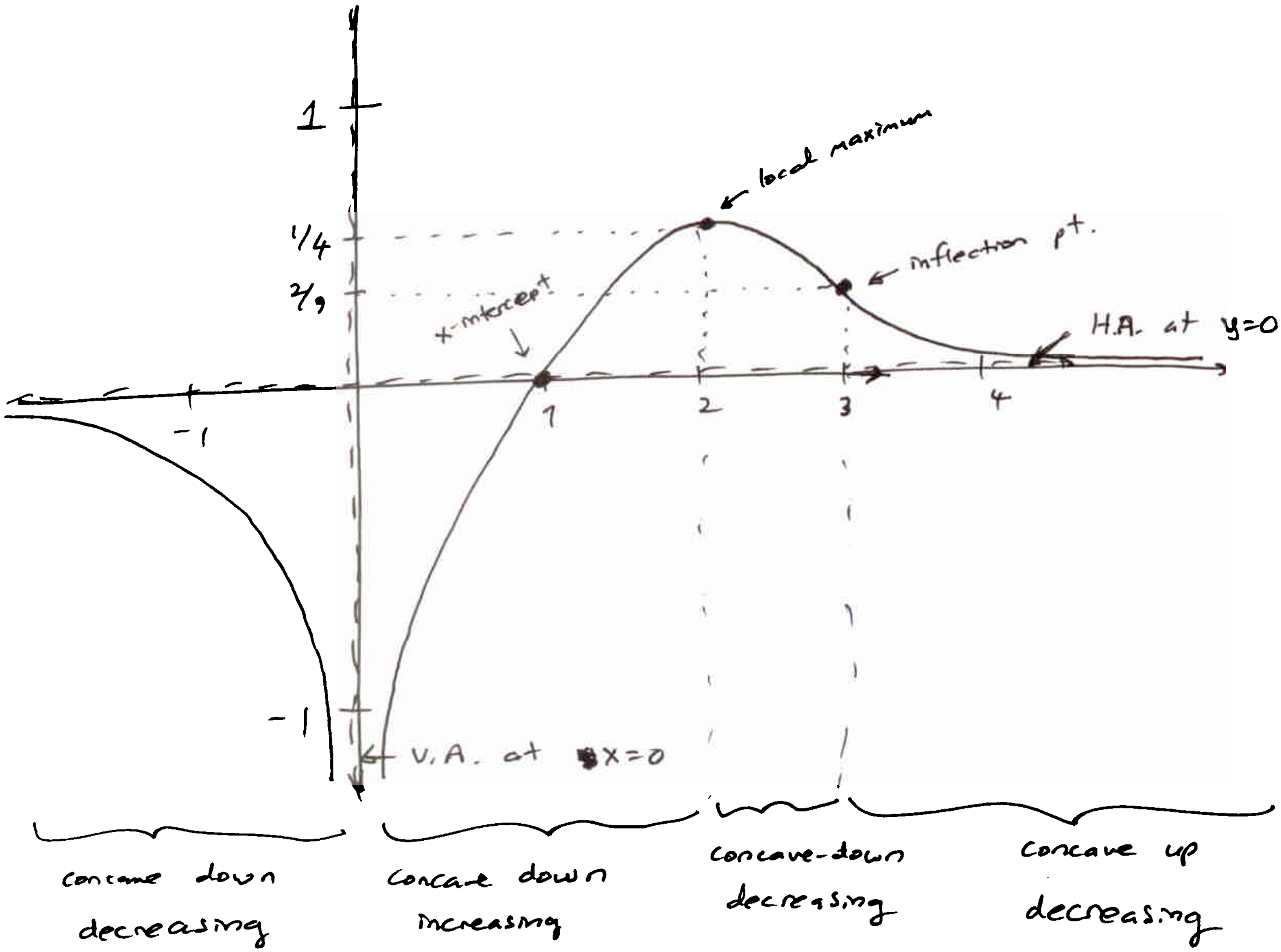
$f(0)$ DNE

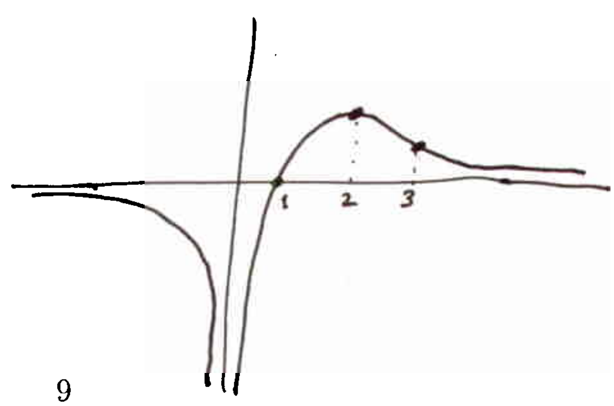
$x=0$ V.A.

$f(3) = \frac{3-1}{3^2} = \frac{2}{9}$

$\boxed{\text{inflection point at } (3, 2/9)}$

(e) (6 pts) Sketch the graph of $f(x)$.





7. (10 pts.) Solve the following initial value problem:

$$\frac{dy}{dx} = \frac{x(1-x^2)}{y^2}, \quad y(0) = 2$$

$$\int y^2 \, dy = \int (x - x^3) \, dx$$

$$\frac{y^3}{3} = \frac{x^2}{2} - \frac{x^4}{4} + C$$

$$y(x) = \sqrt[3]{\frac{3}{2}x^2 - \frac{3}{4}x^4 + C}$$

$$y(0) = \sqrt[3]{0 - 0 + C} = \sqrt[3]{C} = 2$$

$$C = 2^3$$

$$C = 8$$

$$y(x) = \sqrt[3]{\frac{3}{2}x^2 - \frac{3}{4}x^4 + 8}$$