

6.1.4. Let $h = 0.25$ and

$$\int_1^2 \frac{dx}{x} = \frac{0.25}{3} [1 + 4(1.25)^{-1} + 2(1.5)^{-1} + 4(1.75)^{-1} + (2)^{-1}] = 0.6933$$

Error = $-\frac{(b-a)}{180} h^4 f^{(4)}$. Now $f = x^{-1}$, $f' = -x^{-2}$, $f'' = 2x^{-3}$, $f''' = -6x^{-4}$, and

$f^{(4)} = 24x^{-5}$. Hence $|f^{(4)}| \leq 24$ and $|\text{error}| \leq \frac{(.25)^4(24)}{(180)} = 5.2 \times 10^{-4}$ is our bound.

Notice that $\ln 2 = 0.693147$ so that the error is really 1.5×10^{-4} which is less than the bound:

6.1.7. We are given $\int_a^b f(x) dx = S_{n-1} + Ch^4$. Consider with $2(n-1)$ so h is replaced by $h/2$,

$$\int_a^b f(x) dx = S_{2(n-1)} + \frac{Ch^4}{16}$$

Eliminating the Ch^4 terms, we have

$$15 \int_a^b f(x) dx = 16S_{2(n-1)} - S_{n-1}$$

so that

$$\int_a^b f(x) dx = \frac{16}{15} S_{2(n-1)} - \frac{1}{15} S_{n-1}$$

6.1.8. By Taylor series, we have

$$f(a+h) = f + hf' + \frac{1}{2}h^2 f'' + \frac{1}{6}h^3 f''' + \frac{1}{4!}h^4 f^{(4)} + \dots$$

$$f(a+2h) = f + 2hf' + 2h^2 f'' + \frac{4}{3}h^3 f''' + \frac{2^4}{4!}h^4 f^{(4)} + \dots$$

$$f(a+3h) = f + 3hf' + \frac{9}{2}h^2 f'' + \frac{9}{2}h^3 f''' + \frac{3^4}{4!}h^4 f^{(4)} + \dots$$

Hence, we obtain

$$f + 3f(a+h) + 3f(a+2h) + f(a+3h) = 8f + 12hf' + 12h^2 f'' + 9h^3 f''' + \frac{11}{2}h^4 f^{(4)} + \dots$$

and

$$\begin{aligned} \frac{3h}{8} [f + 3f(a+h) + 3f(a+2h) + f(a+3h)] &= 3hf + \frac{9}{2}h^2 f' + \frac{9}{2}h^3 f'' + \frac{27}{8}h^4 f''' \\ &\quad + \frac{33}{16}h^5 f^{(4)} + \dots \end{aligned}$$

Letting $3h$ replace h in (2) of Section 4.2, we have

$$\int_a^{a+3h} f(x) dx = 3hf + \frac{9}{2}h^2 f' + \frac{9}{2}h^3 f'' + \frac{27}{8}h^4 f''' + \frac{81}{40}h^5 f^{(4)} + \dots$$

Consequently,

$$\int_a^{a+3h} f(x) dx = \frac{3h}{8} [f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)] - \frac{3}{80}h^5 f^{(4)} + \dots$$

The error term is $-\frac{3}{80}h^5 f^{(4)}(\xi)$ for some ξ between a and $a+3h$. Both Simpson's $\frac{1}{3}$ -rule and $\frac{3}{8}$ -rule have error terms $\mathcal{O}(h^5)$. Since the Simpson's $\frac{3}{8}$ -rule requires one more function evaluation than the Simpson's $\frac{1}{3}$ -rule for an approximation of the same order of accuracy, it is clearly less desirable.

6.2.1. $a = 0, b = 2, f(x) = e^{-x^2}$

$$\int_0^2 \exp(-x^2) dx \approx \exp\left[-\left(-\sqrt{\frac{1}{3}} + 1\right)^2\right] + \exp\left[-\left(\sqrt{\frac{1}{3}} + 1\right)^2\right] \approx 0.91949$$

6.2.2. Only part (a) will be given:

f	LHS	RHS
1	2	$= 1 + 1$
x	0	$= -\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{3}}$
x^2	$\frac{2}{3}$	$= \frac{1}{3} + \frac{1}{3}$
x^3	0	$= -\frac{1}{3}\sqrt{\frac{1}{3}} + \frac{1}{3}\sqrt{\frac{1}{3}}$
x^4	$\frac{2}{5}$	$\neq \frac{1}{9} + \frac{1}{9}$

6.2.3. LHS = $\int_{-1}^1 f(x) dx$ RHS = $\frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right)$

f	LHS	RHS
1	2	$= \frac{5}{9} + \frac{8}{9} + \frac{5}{9}$
x	0	$= -\frac{5}{9}\sqrt{\frac{3}{5}} + 0 + \frac{5}{9}\sqrt{\frac{3}{5}}$
x^2	$\frac{2}{3}$	$= \frac{1}{3} + 0 + \frac{1}{3}$
x^3	0	$= -\frac{1}{3}\sqrt{\frac{3}{5}} + 0 + \frac{1}{3}\sqrt{\frac{3}{5}}$
x^4	$\frac{2}{5}$	$= \frac{1}{5} + 0 + \frac{1}{5}$
x^5	0	$= -\frac{1}{9}\sqrt{\frac{3}{5}} + 0 + \frac{1}{9}\sqrt{\frac{3}{5}}$
x^6	$\frac{2}{7}$	$\neq \frac{3}{25} + 0 + \frac{3}{25}$

6.2.5. Let LHS = $\int_{-1}^1 f(x) dx$ and RHS = $\alpha f\left(-\frac{1}{2}\right) + \beta f(0) + \gamma f\left(\frac{1}{2}\right)$.

$f(x)$	LHS	RHS
1	2	$\alpha + \beta + \gamma$
x	0	$-\frac{1}{2}\alpha + \frac{1}{2}\gamma$
x^2	$\frac{2}{3}$	$\frac{1}{4}\alpha + \frac{1}{4}\gamma$

Hence for LHS = RHS, we have $\alpha = \gamma = \frac{4}{3}, \beta = -\frac{2}{3}$.

6.2.7. Let LHS = $\int_a^{a+h} f(x) dx$ and RHS = $\alpha f(a) + \beta f(a+h) + \gamma f(a+2h)$.

$f(x)$	LHS	RHS
1	h	$\alpha + \beta + \gamma$
$x - a$	$\frac{h^2}{2}$	$\beta h + \gamma 2h$
$(x - a)^2$	$\frac{h^3}{3}$	$\beta h^2 + \gamma (2h)^2$

Letting LHS = RHS, we have

$$\begin{cases} \alpha + \beta + \gamma = h \\ \beta + 2\gamma = \frac{h}{2} \\ \beta + 4\gamma = \frac{h}{3} \end{cases}$$

Hence, $\beta = \frac{2h}{3}$, $\gamma = -\frac{h}{12}$, $\alpha = \frac{5h}{12}$.

6.2.12. LHS = $\int_{-2}^{+2} f(x) dx$ RHS = $Af(-1) + Bf(0) - Cf(+1)$

f	LHS	RHS
1	4	$= A + B + C$
x	0	$= -A + C$
x^2	8	$= A + C$
x^3	0	$= -A + C$
x^4	$\frac{128}{3}$	$\neq A + C$

$A = 4, B = -4, C = 4$

Yes. Exact for polynomials of degree ≤ 3 .

6.2.13. Let LHS = $\int_{-h}^h f(x) dx$, RHS = $Af(-h) + Bf(0) + Cf(h) - hDf'(h)$

f	f'	LHS	RHS
1	0	$2h$	$= A + B + C$
x	1	0	$= -hA + hC - hD$
x^2	$2x$	$\frac{2}{3}h^3$	$= h^2A + h^2C - 2h^2D$
x^3	$3x^2$	0	$= -h^3A + h^3C - 3h^3D$

$$A + B + C = 2h$$

$$\begin{cases} A - C + D = 0 \\ A + C - 2D = \frac{2}{3}h \\ A - C + 3D = 0 \end{cases}$$

$$\begin{cases} 2A - D = \frac{2}{3}h \\ 2A + D = \frac{2}{3}h \end{cases}$$

$$A = \frac{h}{3}, \quad D = 0$$

$$C = \frac{h}{3}, \quad B = \frac{4}{3}h$$

6.2.16. Let $t = 2x - 1$. Then $x = 1$ corresponds to $t = 1$, and $x = 0$ corresponds to $t = -1$. Furthermore, $dx = (1/2)dt$. Thus

$$\begin{aligned} \int_0^1 f(x) dx &= \frac{1}{2} \int_{-1}^1 f\left(\frac{1}{2}(t+1)\right) dt \\ &\approx \frac{1}{2} \sum_{i=1}^n w_i f\left(\frac{1}{2}x_i + \frac{1}{2}\right) \end{aligned}$$