

---

# Difference Equations

Caner Kazanci

Math 2700  
University of Georgia

# *Death - Birth Processes*

---

$N_t$  = Number of Bacteria at time  $t$  (State)

$$N_{t+1} = N_t - \text{Deaths} + \text{Births}$$

# Death - Birth Processes

---

$N_t$  = Number of Bacteria at time  $t$  (State)

$$\begin{aligned} N_{t+1} &= N_t - \text{Deaths} + \text{Births} \\ &= N_t - d N_t + b N_t \end{aligned}$$

# Death - Birth Processes

---

$N_t$  = Number of Bacteria at time  $t$  (State)

$$\begin{aligned}N_{t+1} &= N_t - \text{Deaths} + \text{Births} \\ &= N_t - d N_t + b N_t \\ &= (1 - d + b) N_t \\ &= r N_t\end{aligned}$$

# Death - Birth Processes

---

$N_t$  = Number of Bacteria at time  $t$  (State)

$$\begin{aligned}N_{t+1} &= N_t - \text{Deaths} + \text{Births} \\ &= N_t - d N_t + b N_t \\ &= (1 - d + b) N_t \\ &= r N_t\end{aligned}$$

$r = 1 - d + b$  : Net growth rate

# Growth

---

Unlimited resources (food, space), no death ( $d = 0$ )

eg. Let the birth rate be equal to 1 ( $b = 1$ )

# Growth

---

Unlimited resources (food, space), no death ( $d = 0$ )  
eg. Let the birth rate be equal to 1 ( $b = 1$ )

$$r = 1 - d + b = 1 - 0 + 1 = 2$$

**Model :**  $N_{t+1} = 2 N_t$

# Growth

---

Unlimited resources (food, space), no death ( $d = 0$ )  
eg. Let the birth rate be equal to 1 ( $b = 1$ )

$$r = 1 - d + b = 1 - 0 + 1 = 2$$

**Model :**  $N_{t+1} = 2 N_t$

Time units : 0 → 1 → 2 → 3 → 4 → 5

# Bacteria : 1 → 2 → 4 → 8 → 16 → 32

**Question:** When will the bacteria be equal to  $10^9$ ?

# Growth

Unlimited resources (food, space), no death ( $d = 0$ )  
eg. Let the birth rate be equal to 1 ( $b = 1$ )

$$r = 1 - d + b = 1 - 0 + 1 = 2$$

**Model :**  $N_{t+1} = 2 N_t$

Time units :  $0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$

# Bacteria :  $1 \rightarrow 2 \rightarrow 4 \rightarrow 8 \rightarrow 16 \rightarrow 32$

**Question:** When will the bacteria be equal to  $10^9$ ?

$$2^n = 10^9$$

# Growth

Unlimited resources (food, space), no death ( $d = 0$ )  
eg. Let the birth rate be equal to 1 ( $b = 1$ )

$$r = 1 - d + b = 1 - 0 + 1 = 2$$

**Model :**  $N_{t+1} = 2 N_t$

Time units : 0 → 1 → 2 → 3 → 4 → 5

# Bacteria : 1 → 2 → 4 → 8 → 16 → 32

**Question:** When will the bacteria be equal to  $10^9$ ?

$$2^n = 10^9 = (10^3)^3 \approx (2^{10})^3$$

# Growth

Unlimited resources (food, space), no death ( $d = 0$ )  
eg. Let the birth rate be equal to 1 ( $b = 1$ )

$$r = 1 - d + b = 1 - 0 + 1 = 2$$

**Model :**  $N_{t+1} = 2 N_t$

Time units :  $0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$

# Bacteria :  $1 \rightarrow 2 \rightarrow 4 \rightarrow 8 \rightarrow 16 \rightarrow 32$

**Question:** When will the bacteria be equal to  $10^9$ ?

$$2^n = 10^9 = (10^3)^3 \approx (2^{10})^3 = 2^{30}$$

**Answer:**  $n = 30$  time units

# Death

---

Birth rate ( $b$ ) = 0

Model (recursive definition):

$$N_{t+1} = (1 - d + b) N_t$$

# Death

---

Birth rate ( $b$ ) = 0

Model (recursive definition):

$$N_{t+1} = (1 - d + b) N_t = (1 - d) N_t$$

Model (explicit definition):

# Death

---

Birth rate  $(b) = 0$

Model (recursive definition):

$$N_{t+1} = (1 - d + b) N_t = (1 - d) N_t$$

Model (explicit definition):

$$N_t = (1 - d)^t N_0$$

Since  $1 - d < 1$ ,  $N_t \rightarrow$

# Death

---

Birth rate ( $b$ ) = 0

Model (recursive definition):

$$N_{t+1} = (1 - d + b) N_t = (1 - d) N_t$$

Model (explicit definition):

$$N_t = (1 - d)^t N_0$$

Since  $1 - d < 1$ ,  $N_t \rightarrow 0$ . Sooner or later all bacteria die.

# Death

---

Birth rate ( $b$ ) = 0

Model (recursive definition):

$$N_{t+1} = (1 - d + b) N_t = (1 - d) N_t$$

Model (explicit definition):

$$N_t = (1 - d)^t N_0$$

Since  $1 - d < 1$ ,  $N_t \rightarrow 0$ . Sooner or later all bacteria die.

General Solution:

$$N_t = (1 - d + b)^t N_0 = r^t N_0$$

Outcomes:

# Death

Birth rate ( $b$ ) = 0

Model (recursive definition):

$$N_{t+1} = (1 - d + b) N_t = (1 - d) N_t$$

Model (explicit definition):

$$N_t = (1 - d)^t N_0$$

Since  $1 - d < 1$ ,  $N_t \rightarrow 0$ . Sooner or later all bacteria die.

**General Solution:**

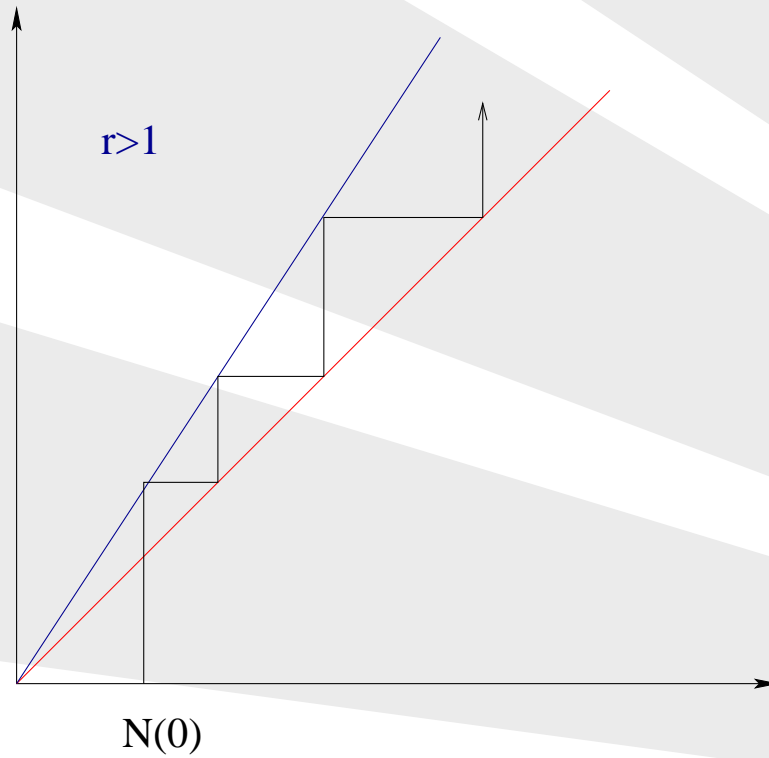
$$N_t = (1 - d + b)^t N_0 = r^t N_0$$

**Outcomes:** Exponential growth, exponential decay, constant solution.

# Cobwebs

Net Growth rate ( $r > 1$ )

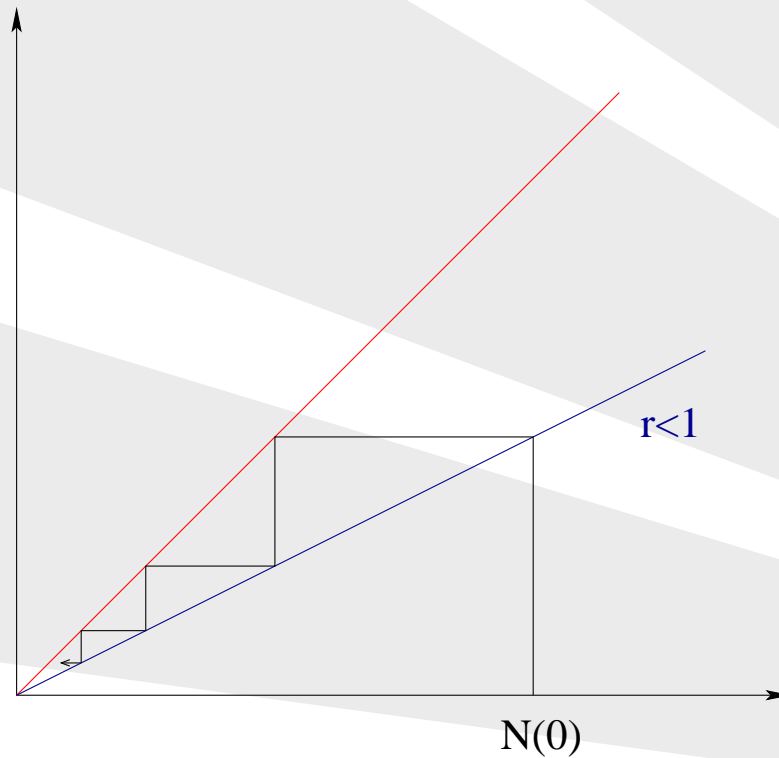
Exponential Growth



# Cobwebs

Net Growth rate ( $r < 1$ )

Exponential Decay



# *Variable growth rate*

---

Constant growth rate is not realistic.

Next Step: Growth rate depends on population

$$r = r(N)$$

# Variable growth rate

---

Constant growth rate is not realistic.

Next Step: Growth rate depends on population

$$r = r(N)$$
$$r(N) = c \left( 1 - \frac{N}{K} \right)$$

$K$  : Carrying capacity

# Variable growth rate

Constant growth rate is not realistic.

Next Step: Growth rate depends on population

$$r = r(N)$$
$$r(N) = c \left( 1 - \frac{N}{K} \right)$$

$K$  : Carrying capacity

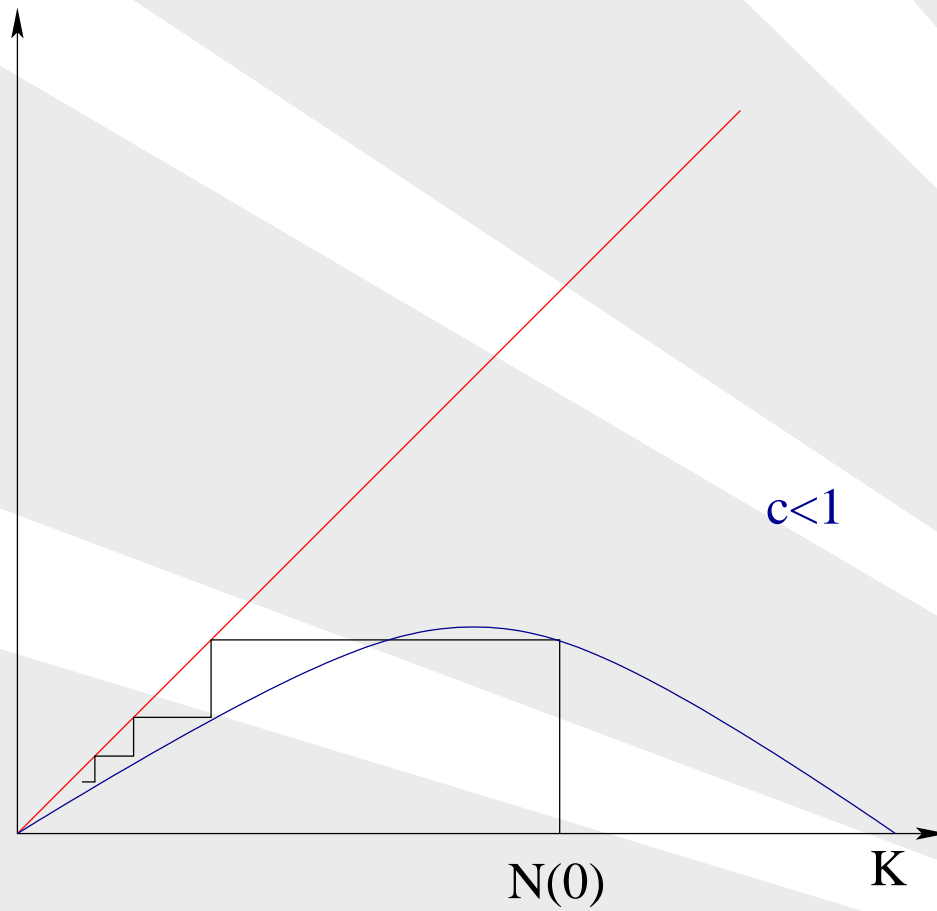
Let

$$n_t = \frac{N_t}{K}$$

Then

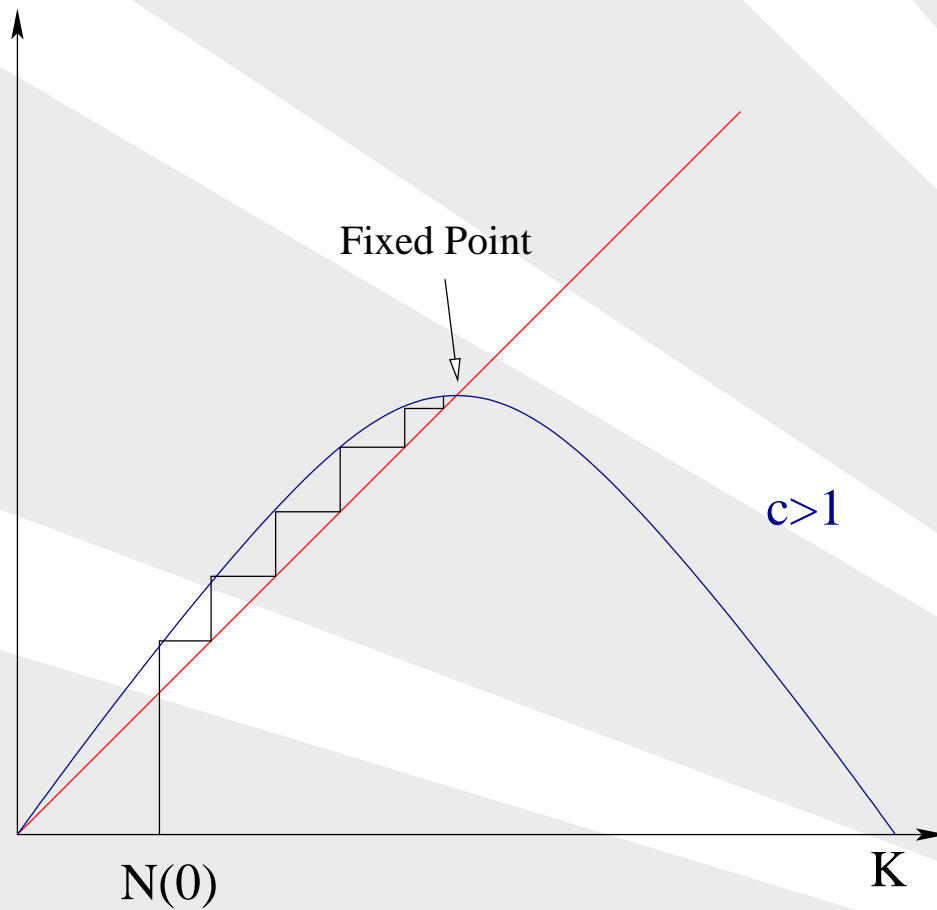
$$n_{t+1} = c(1 - n_t)n_t$$

# Cobweb



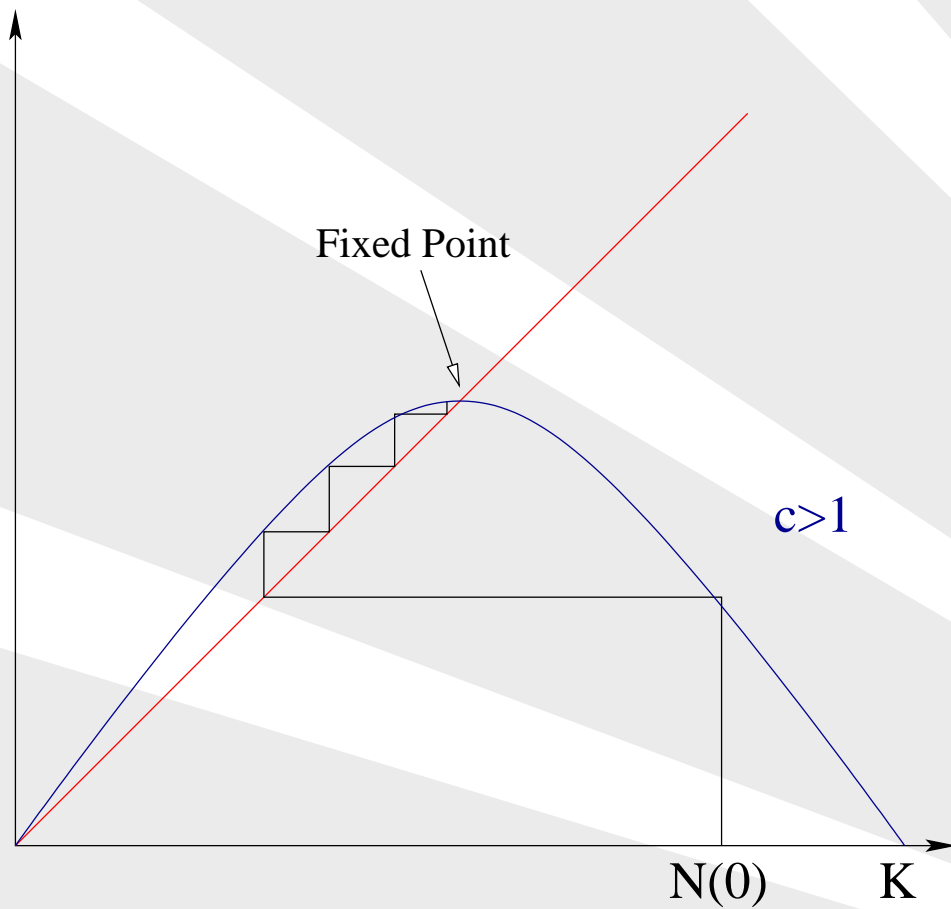
$$n_{t+1} = c(1 - n_t)n_t \quad c < 1$$

# Cobweb



$$n_{t+1} = c(1 - n_t)n_t \quad c > 1$$

# Cobweb



$$n_{t+1} = c(1 - n_t)n_t \quad c > 1$$

# *Discrete time population models*

---

$$N_{t+1} = r(N_t) N_t$$

$r(N)$  = New growth rate

# *Discrete time population models*

---

$$N_{t+1} = r(N_t) N_t$$

$r(N)$  = New growth rate

Example:

$$r(N) = c N (K - N)$$

# Discrete time population models

$$N_{t+1} = r(N_t) N_t$$

$r(N)$  = New growth rate

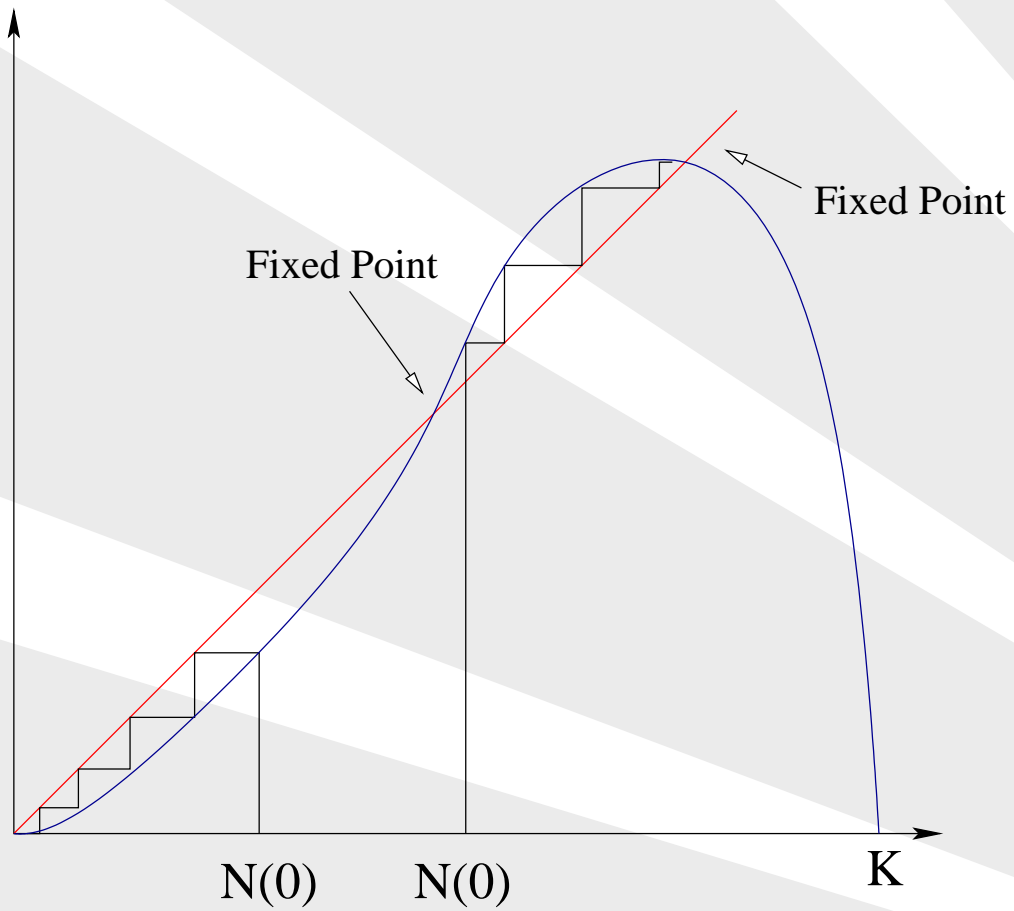
Example:

$$r(N) = c N (K - N)$$

Rescale by  $K$ :

$$n_{t+1} = c (1 - n_t) n_t^2, \quad n = \frac{N}{K}$$

# Cobweb



$$n_{t+1} = c(1 - n_t)n_t^2$$

# *From Discrete to Continuous*

---

$$N_{t+1} = (1 - d + b) N_t(t)$$

# *From Discrete to Continuous*

---

$$N_{t+1} = (1 - d + b) N_t(t)$$

$$\frac{N_{t+1} - N_t}{\Delta t} = (b - d) N_t$$

# From Discrete to Continuous

---

$$N_{t+1} = (1 - d + b) N_t(t)$$

$$\frac{N_{t+1} - N_t}{\Delta t} = (b - d) N_t$$
$$\frac{dN(t)}{dt} \approx (b - d) N(t) = f(N(t))$$

General case?

# From Discrete to Continuous

$$N_{t+1} = (1 - d + b) N_t(t)$$

$$\frac{N_{t+1} - N_t}{\Delta t} = (b - d) N_t$$
$$\frac{dN(t)}{dt} \approx (b - d) N(t) = f(N(t))$$

General case? **Fixed points?**

# From Discrete to Continuous

$$N_{t+1} = (1 - d + b) N_t(t)$$

$$\frac{N_{t+1} - N_t}{\Delta t} = (b - d) N_t$$
$$\frac{dN(t)}{dt} \approx (b - d) N(t) = f(N(t))$$

General case? **Fixed points?**

Discrete time fixed points:  $N_{t+1} = N_t = N^*$

# From Discrete to Continuous

$$N_{t+1} = (1 - d + b) N_t(t)$$

$$\frac{N_{t+1} - N_t}{\Delta t} = (b - d) N_t$$
$$\frac{dN(t)}{dt} \approx (b - d) N(t) = f(N(t))$$

General case? **Fixed points?**

Discrete time fixed points:  $N_{t+1} = N_t = N^*$

Continuous time fixed points:  $f(N^*) = 0$

# *Long term behavior*

---

Fixed growth rate model:

$$N_{t+1} = r N_t$$

# *Long term behavior*

---

Fixed growth rate model:

$$N_{t+1} = r N_t$$

With carrying capacity:

$$N_{t+1} = c N_t \left(1 - \frac{N_t}{K}\right)$$

# Long term behavior

---

Fixed growth rate model:

$$N_{t+1} = r N_t$$

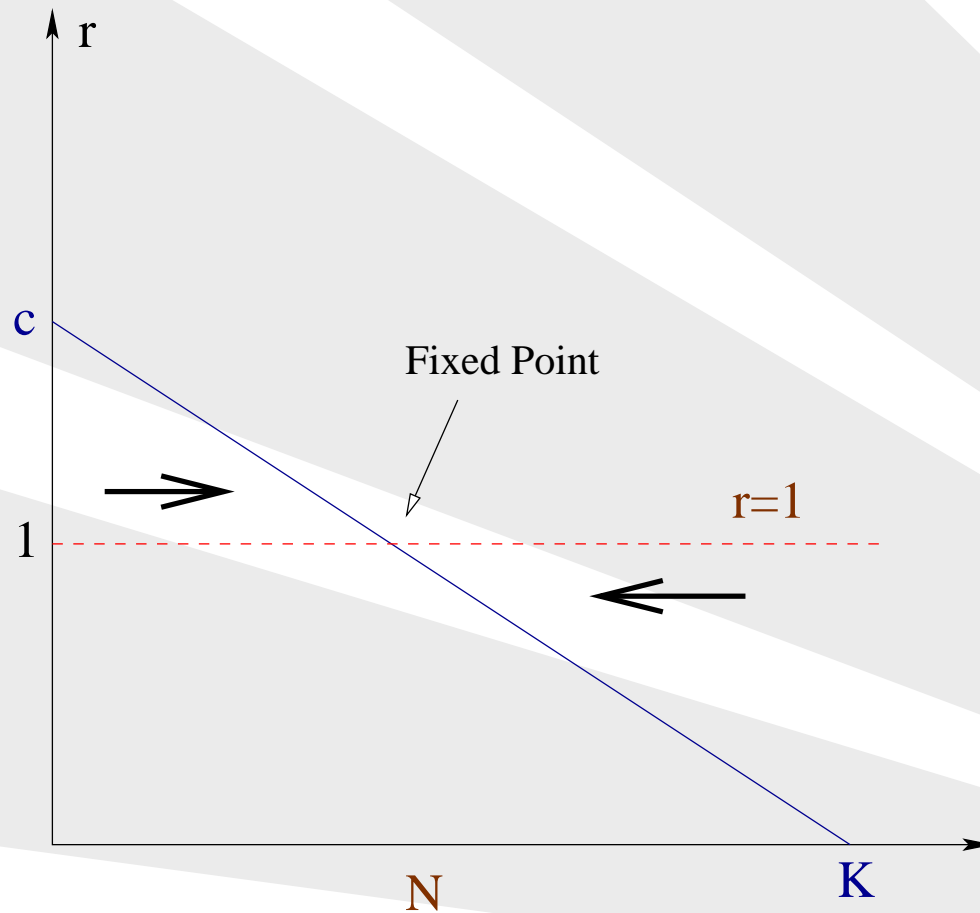
With carrying capacity:

$$N_{t+1} = c N_t \left(1 - \frac{N_t}{K}\right)$$

Is the fixed point  $N^* = K$  **stable**?

# Stability

eg.  $r(N) = c\left(1 - \frac{N}{K}\right)$



# Stability

eg.  $r(n) = cn(1 - n)$

