

A Slightly Different Approach to Network Environ Analysis

Caner Kazancı

University of Georgia

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Outline

Model Network
Structure
Contents of NEA
Mass-Action Type
Interactions
Rate of Change of
Storage Values
Construction of ODE
ODE System
Storage Analysis
Flow Analysis
Throughflow Analysis
Throughflow Analysis
Non-Steady-State
Throughflow Analysis
Utility Analysis

Model Network Structure
Contents of NEA
Mass-Action Type Interactions
Rate of Change of Storage Values
Construction of ODE
ODE System
Storage Analysis
Flow Analysis
Throughflow Analysis
Throughflow Analysis
Non-Steady-State Throughflow Analysis
Utility Analysis

Model Network Structure

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Contents of NEA
Mass-Action Type

Interactions
Rate of Change of
Storage Values

Construction of ODE
ODE System

Storage Analysis

Flow Analysis

Throughflow Analysis

Throughflow Analysis

Non-Steady-State

Throughflow Analysis

Utility Analysis

- A network with n compartments:

$$x_1, x_2, \dots, x_n$$

- Linear interactions among compartments:

$$x_j \rightarrow x_i, \quad 1 \leq i, j \leq n$$

- Inputs to and outputs from individual compartments:

$$x_i \rightarrow \odot, \quad \odot \rightarrow x_j, \quad 1 \leq i, j \leq n$$

Contents of NEA

Model Network
Structure

Contents of NEA
Mass-Action Type
Interactions

Rate of Change of
Storage Values

Construction of ODE
ODE System

Storage Analysis

Flow Analysis

Throughflow Analysis

Throughflow Analysis

Non-Steady-State

Throughflow Analysis

Utility Analysis

■ Pathway analysis

- Number of possible pathways among compartments.

■ Storage Analysis

- Mapping of input values to storage values at steady state.

■ Flow Analysis

- Mapping of input values to sum of flows from/to each compartment.

■ Utility Analysis

- Analysis of $+/-$ relationships among compartments.

Mass-Action Type Interactions

- Partial turnover coefficient of $x_j \rightarrow x_i$ is denoted as c_{ij} (\sim community matrix).
- Then the actual flow intensity of $x_j \rightarrow x_i$ is $c_{ij}x_j$.
- Output flow intensity of $x_j \rightarrow \odot$ is $c_{0j}x_j = y_j$.
- Input flow intensity of $\odot \rightarrow x_j$ is just $z_j = c_{j0}$.
- Turnover rate is defined for each compartment x_k as follows:

$$\tau_k = \sum_{i=0, i \neq k}^n c_{ik}, \quad k = 1, \dots, n$$

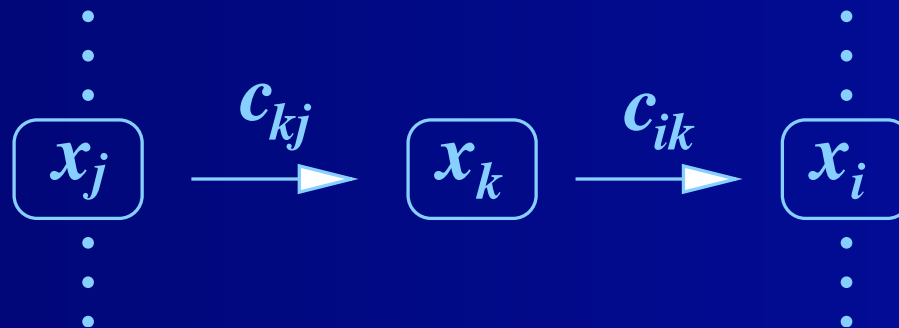
Model Network
Structure
Contents of NEA
Mass-Action Type
Interactions
Rate of Change of
Storage Values
Construction of ODE
ODE System
Storage Analysis
Flow Analysis
Throughflow Analysis
Throughflow Analysis
Non-Steady-State
Throughflow Analysis
Utility Analysis

Rate of Change of Storage Values

The rate of change of compartment x_k can be formulated as:

$$\frac{dx_k}{dt} = \underbrace{\sum_{j=0, j \neq k}^n x_j c_{kj}}_{\text{inflow}} - x_k \underbrace{\sum_{i=0, i \neq k}^n c_{ik}}_{\text{outflow}}, \quad 1 \leq k \leq n$$

where $x_0 = 1$.



Construction of ODE

The rate of change of compartment x_k can be formulated as:

$$\frac{dx_k}{dt} = -x_k \underbrace{\sum_{i=0, i \neq k}^n c_{ik}}_{\text{outflow}} + \underbrace{\sum_{j=0, j \neq k}^n x_j c_{kj}}_{\text{inflow}}, \quad 1 \leq k \leq n$$

where $x_0 = 1$. Then

$$\frac{dx_k}{dt} = -x_k \tau_k + \sum_{j=0, j \neq k}^n x_j c_{kj}, \quad 1 \leq k \leq n$$

Using vector notation, we get

$$\frac{dx_k}{dt} = [c_{k1} \ c_{k2} \ \cdots \ -\tau_k \ \cdots \ c_{kn}] [x_1 \ x_2 \ \cdots \ x_n]^T + c_{k0}$$

Model Network
Structure
Contents of NEA
Mass-Action Type
Interactions
Rate of Change of
Storage Values
Construction of ODE
ODE System
Storage Analysis
Flow Analysis
Throughflow Analysis
Throughflow Analysis
Non-Steady-State
Throughflow Analysis
Utility Analysis

ODE System

- Model Network Structure
- Contents of NEA Mass-Action Type Interactions
- Rate of Change of Storage Values
- Construction of ODE
- ODE System**
- Storage Analysis
- Flow Analysis
- Throughflow Analysis
- Throughflow Analysis
- Non-Steady-State Throughflow Analysis
- Utility Analysis

Let

$$x = [x_1 \ x_2 \ \cdots \ x_n]^T, \quad z = [z_1 \ z_2 \ \cdots \ z_n]^T$$

Then

$$\dot{x} = Cx + z$$

where

$$C = \begin{bmatrix} -\tau_1 & c_{12} & \cdots & c_{1n} \\ c_{21} & -\tau_2 & \cdots & c_{1(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{(n-1)1} & \cdots & -\tau_n \end{bmatrix}$$

Storage Analysis

Storage Analysis immediately follows from this ODE system:

$$\dot{x} = C x + z$$

Assuming the system is at steady state ($\dot{x}^* = 0$), we get

$$\begin{aligned} -C x^* &= z \\ x^* &= (-C^{-1}) z \\ x^* &= S z \end{aligned}$$

The mapping represented by the matrix S

$$S : z \rightarrow x^*$$

maps *inputs* into *steady-state storage values*.

- Model Network Structure
- Contents of NEA Mass-Action Type Interactions
- Rate of Change of Storage Values
- Construction of ODE ODE System
- Storage Analysis**
- Flow Analysis
- Throughflow Analysis
- Throughflow Analysis Non-Steady-State
- Throughflow Analysis
- Utility Analysis

Flow Analysis

- Model Network Structure
- Contents of NEA Mass-Action Type Interactions
- Rate of Change of Storage Values
- Construction of ODE ODE System
- Storage Analysis
- Flow Analysis
- Throughflow Analysis
- Throughflow Analysis Non-Steady-State
- Throughflow Analysis
- Utility Analysis

To accommodate flow analysis we define the flow matrix F as follows:

$$F := \begin{cases} f_{ij} & = & c_{ij}x_j^* \\ f_{ii} & = & -\tau_i x_i^* = -T_i \end{cases}$$

where x^* is the state of the system at steady-state, i.e.,

$$C x^* + z = 0$$

The equality below follows from the definitions of F and C .

$$F \mathbf{1} = C x^*$$

Throughflow Analysis

We define the matrix B as follows:

$$B := \begin{cases} b_{ij} & = c_{ij}/\tau_j = f_{ij}/T_j \\ b_{ii} & = -1 \end{cases}$$

Then by definition

$$F \mathbf{1} = B T$$

Note that at steady state we have

$$F \mathbf{1} = C x^* = -z$$

- Model Network Structure
- Contents of NEA
- Mass-Action Type Interactions
- Rate of Change of Storage Values
- Construction of ODE
- ODE System
- Storage Analysis
- Flow Analysis
- Throughflow Analysis**
- Throughflow Analysis
- Non-Steady-State
- Throughflow Analysis
- Utility Analysis

Throughflow Analysis

- Model Network Structure
- Contents of NEA
- Mass-Action Type Interactions
- Rate of Change of Storage Values
- Construction of ODE ODE System
- Storage Analysis
- Flow Analysis
- Throughflow Analysis
- Throughflow Analysis**
- Non-Steady-State Throughflow Analysis
- Utility Analysis

Combining

$$F \mathbf{1} = B T$$

with

$$F \mathbf{1} = C x^* = -z$$

we get

$$\begin{aligned} -z &= B T \\ \underbrace{(-B^{-1})}_{=N} z &= T \end{aligned}$$

The mapping represented by the matrix

$$N : z \rightarrow T$$

maps inputs to throughflows.

Non-Steady-State Throughflow Analysis

Is it possible to derive a map from inputs to throughflows

$$\bar{N} : z \rightarrow T$$

in the non-steady-state case?

Since throughflows (T) and storages (x) will be varying in time, such a map would probably be a *differential map*.

$$\begin{aligned}\dot{x} &= Cx + z \\ &= BT \cdot x + z\end{aligned}$$

where “ \cdot ” denotes componentwise vector multiplication. Rearranging terms, we get

$$\underbrace{\left(\frac{d}{dt} - BT\right)}_{=\bar{N}} \cdot x = z$$

Model Network
Structure
Contents of NEA
Mass-Action Type
Interactions
Rate of Change of
Storage Values
Construction of ODE
ODE System
Storage Analysis
Flow Analysis
Throughflow Analysis
Throughflow Analysis
Non-Steady-State
Throughflow Analysis
Utility Analysis

Utility Analysis

- Model Network Structure
- Contents of NEA Mass-Action Type Interactions
- Rate of Change of Storage Values
- Construction of ODE ODE System
- Storage Analysis
- Flow Analysis
- Throughflow Analysis
- Throughflow Analysis
- Non-Steady-State Throughflow Analysis
- Utility Analysis

Utility Analysis is based on the matrix

$$U = (I - D)^{-1} \quad \text{where} \quad D = B - B'$$

The matrix B' is defined similarly to B as follows:

$$B' := \begin{cases} b'_{ij} & = c_{ji}/\tau_j = f_{ji}/T_j \\ b'_{ii} & = -1 \end{cases}$$

Or similarly:

$$D := \begin{cases} d_{ij} & = (c_{ij} - c_{ji})/\tau_i = (f_{ij} - f_{ji})/T_i \\ d_{ii} & = 0 \end{cases}$$