ENGR 6101: COMPUTATIONAL ENGINEERING

Solution Set 1 *(due Monday in class, 8/31)*

Show all your work on all problems. Correct answers without enough work will be graded as 0/3.

Questions:

1. (3 pts.) A computer with 4 digit precision would recognize the given values as follows:

$$\begin{aligned} \pi &\rightarrow 0.3142\times 10^{1} \\ 12347.89 &\rightarrow 0.1235\times 10^{5} \\ 0.0001231234 &\rightarrow 0.1231\times 10^{-3} \\ 1 &\rightarrow 0.1000\times 10^{1} \end{aligned}$$

We can emulate the algebraic operations on such a computer by doing the same operation on a higher precision machine (eg. a calculator,matlab, etc.) and than decrease the precision as shown above (rounding to the nearest decimal). But, we have to round after each basic operation, such as sum, product, square root, etc.

(a) Consider the following expression:

$$\sqrt{x}-3$$

On such a computer, compute the expression above for $x = 9.01$. [1]

SOLUTION. $\sqrt{9.01} = 3.001666203960727... \rightarrow 0.3002\times 10^{1}$. So the answer is

$$0.3002\times 10^{1} - 0.3000\times 10^{1} \approx 0.0002\times 10^{1} = 0.2000\times 10^{-2}$$

(b) Note that

$$\sqrt{x}-3 = \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}+3} = \frac{x-9}{\sqrt{x}+3}$$

On the same computer, evaluate the last expression above, again for $x = 9.01$.

SOLUTION. $9.01-9 = 0.01 = 0.1000\times 10^{-1}$. $\sqrt{9.01}\approx 0.3002\times 10^{1}$. $\sqrt{9.01}+3\approx 0.6002\times 10^{1}$.

$$\frac{0.1000\times 10^{-1}}{0.6002\times 10^{1}} = 0.001666203960727 \rightarrow 0.1666\times 10^{-2}$$

(c) Use a higher precision machine (at least 8 digits) to compute $\sqrt{9.01}-3$, and write down this result. Assuming that this results is "true", compute the absolute and relative errors (accurate upto at least 6 digits) for the results you obtained in part (a) and (b).

SOLUTION. Exact result is $0.166620396\times 10^{-2}$.
Part (a): Absolute error is $|.002 - 0.00166620396| = 0.33380\times 10^{-3}$. Relative error is:

$$\left|\frac{.002-0.00166620396}{0.00166620396}\right| \approx 0.200333$$

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[1] In 1(a) and 1(b), do the operations step by step. For example, in 1(a), first do the square root (on a high precision machine), and then adjust the precision to four digits. Then do the difference (again on a high precision machine), and again limit the precision of the result to four digits.

Part (b): Absolute error is $|.001666 - 0.00166620396| = 0.20396 \times 10^{-6}$. Relative error is:

$$\left| \frac{.001666 - 0.00166620396}{0.00166620396} \right| \approx 0.000122$$

Depending on the precision of your "actual answer", your results might differ one or two digits, which is fine. If this is the case, give yourselves full credit.

2. (3 pts.) Consider the following numerical differentiation formula:

$$f'(x) \approx \frac{\alpha f(x+h) + \beta f(x-2h)}{\gamma h}$$

Find the appropriate values for α, β and γ for this formula to work (using Taylor series expansion). What is the order of your approximation?

SOLUTION. Here are the Taylor series expansions:

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \cdots \\ f(x-2h) &= f(x) - 2hf'(x) + 2h^2 f''(x) + \cdots \end{aligned}$$

Note that

$$f(x+h) - f(x-2h) = 3hf'(x) - \frac{3h^2}{2} f''(x) + \cdots$$

Then $\alpha = 1$, $\beta = -1$ and $\gamma = 3$ are good values for the derivative. Note that

$$\frac{f(x+h) - f(x-2h)}{3h} = f'(x) - \underbrace{\frac{1}{2} h f''(x) + \cdots}_{\text{error term}}$$

The lowest power of h in the error term is 1, therefore this is a first order approximation. If you have computed the error exactly, that is also fine. If this is the case, please give yourselves full credits. However, note that the order should still be the same. Same is valid for the following problem.

3. (3 pts) Consider the following numerical differentiation formula:

$$f'(x) \approx \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}$$

Prove that this is a valid formula (using Taylor's expansion) and find its order.

SOLUTION. Here are the Taylor series expansions:

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f''''(x) + \frac{h^5}{120} f^{(5)}(x) + \cdots \\ f(x+2h) &= f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{4h^3}{3} f'''(x) + \frac{2h^4}{3} f''''(x) + \frac{4h^5}{15} f^{(5)}(x) + \cdots \\ f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f''''(x) - \frac{h^5}{120} f^{(5)}(x) + \cdots \\ f(x-2h) &= f(x) - 2hf'(x) + 2h^2 f''(x) - \frac{4h^3}{3} f'''(x) + \frac{2h^4}{3} f''''(x) - \frac{4h^5}{15} f^{(5)}(x) + \cdots \end{aligned}$$

Multiplying these equations with appropriate coefficients, we get:

$$\begin{aligned}
8f(x+h) &= 8f(x)+8hf'(x)+4h^2f''(x)+\frac{4h^3}{3}f'''(x)+\frac{h^4}{3}f''''(x)+\frac{h^5}{15}f^{(5)}(x)+\cdots \\
-f(x+2h) &= -f(x)-2hf'(x)-2h^2f''(x)-\frac{4h^3}{3}f'''(x)-\frac{2h^4}{3}f''''(x)-\frac{4h^5}{15}f^{(5)}(x)+\cdots \\
-8f(x-h) &= -8f(x)+8hf'(x)-4h^2f''(x)+\frac{4h^3}{3}f'''(x)-\frac{h^4}{3}f''''(x)+\frac{h^5}{15}f^{(5)}(x)+\cdots \\
f(x-2h) &= f(x)-2hf'(x)+2h^2f''(x)-\frac{4h^3}{3}f'''(x)+\frac{2h^4}{3}f''''(x)-\frac{4h^5}{15}f^{(5)}(x)+\cdots
\end{aligned}$$

Adding these equations side by side, we get:

$$-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)=12hf'(x)-\frac{2h^5}{5}f^{(5)}(x)+\cdots$$

which validates that this formula. Note that

$$\frac{-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)}{12h}=f'(x)-\underbrace{\frac{h^4}{30}f^{(5)}(x)+\cdots}_{\text{error term}}$$

The lowest power of h in the error term is 4, therefore this is a fourth order approximation.

4. (3 pts) Consider the following function:

$$f(x)=x^2+sin(x)$$

   (a) Create a table by evaluating this function at $x=0.6$, 0.8, 1, 1.2 and 1.4, accurate upto at least 8 digits[2]. Then compute the numerical approximation of the derivative at $x=1$ using the following methods below:

      i. Forward, use values $f(1)$ and $f(1.2)$.
      ii. Backward, use values $f(1)$ and $f(0.8)$.
      iii. Central, use values $f(0.8)$ and $f(1.2)$.
      iv. New method presented in problem 3, use values $f(0.6)$, $f(0.8)$, $f(1.2)$ and $f(1.4)$.

      SOLUTION. Here is the table:

$$\begin{aligned}
f(.6) &= 0.92464247339503 \\
f(.8) &= 1.35735609089952 \\
f(1) &= 1.84147098480789 \\
f(1.2) &= 2.37203908596722 \\
f(1.4) &= 2.94544972998846
\end{aligned}$$

      Derivatives for each four methods are as follows:

$$\frac{f(1.2)-f(1)}{0.2} = 2.652840505796650$$

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[2] Note that x is in radians, that is, $\sin(2\pi)=1$.

$$\begin{aligned} \frac{f(1)-f(0.8)}{0.2} &= 2.420574469541851 \\ \frac{f(1.2)-f(0.8)}{0.4} &= 2.536707487669250 \\ \frac{-f(1.4)+8f(1.2)-8f(0.8)+f(0.6)}{2.4} &= 2.540273626645071 \end{aligned}$$

Depending on the precision of your "actual answer", your results might differ one or two digits, which is fine. If this is the case, give yourselves full credit. Same is valid for part (b).

(b) Compute the exact value by taking the derivative of this function and evaluating the derivative at $x = 1$ (accurate upto at least 8 digits). Assuming that this results is "true", find the absolute errors for all the methods above.

SOLUTION. Exact derivative:

$$2 + cos(1) = 2.54030230586814$$

Errors for each four methods are as follows:

$$\begin{gathered} 0.11253819992850 \\ 0.11972783632627 \\ 0.00359481819888 \\ 0.0000286792230554944 \end{gathered}$$