ENGR 6101: COMPUTATIONAL ENGINEERING

Problem Set 3 *(due Monday in class, 9/21)*

Questions:

1. Compute the integral

$$\int_0^{\pi/2} \sin(x)\,dx = 1$$

in the following ways:

(a) (1 pts.) Using Trapezoid rule with 1 trapezoid ($x = 0$, $\frac{\pi}{2}$).

SOLUTION.

$$(\frac{a+b}{2})h = \frac{\sin(0)+\sin(\pi/2)}{2}\frac{\pi}{2} = \frac{\pi}{4} \approx 0.7854$$

(b) (1 pts.) Using Trapezoid rule with 2 trapezoids ($x = 0$, $\frac{\pi}{4}$, $\frac{\pi}{2}$).

SOLUTION.

$$\frac{\sin(0)+2\sin(\pi/4)+\sin(\pi/2)}{2}\frac{\pi}{4} = \frac{(\sqrt{2}+1)\pi}{8} \approx 0.9418$$

(c) (2 pts.) Using Richardson's extrapolation with the results in (a) and (b).

SOLUTION. Trapezoid rule is a second order method. Therefore:

$$\frac{4 \cdot 0.9418 - 0.7854}{3} = 1.0023$$

(d) (2 pts.) Using Simpson's rule with 3 data points ($x = 0$, $\frac{\pi}{4}$, $\frac{\pi}{2}$).

SOLUTION.

$$\frac{\sin(0)+4\sin(\pi/4)+\sin(\pi/2)}{6}\frac{\pi}{2} = \frac{(2\sqrt{2}+1)\pi}{12} \approx 1.0023$$

(e) (2 pts.) Write down the exact error for all of the above. Which numerical method gives the most accurate result?

SOLUTION. The exact errors are 0.2146, 0.0519, 0.0023 and 0.0023. Both Simpson's rule, and the two trapezoid rules combined with Richardson extrapolation (also called Romberg) give equally good results.

(f) (2 pts.) Prove that for the following integral

$$\int_0^{\pi/2} f(x)\,dx$$

(c) and (d) will always give the exact same result for any function $f(x)$.

SOLUTION. Simpson's rule:

$$S = \frac{\pi}{12}\left(f(0)+4f(\pi/4)+f(\pi/2)\right)$$

Trapezoid rule using one and two trapezoids:

$$T_1 = \frac{\pi}{4}\left(f(0) + f(\pi/2)\right) \qquad T_2 = \frac{\pi}{8}\left(f(0) + f(\pi/2)\right) + \frac{\pi}{8}\left(f(\pi/2) + f(\pi/4)\right)$$

Combining T_1 and T_2 using Richardson extrapolation, we get

$$R = \frac{4T_2 - T_1}{3} = \frac{\pi}{12}\left(f(0) + 4f(\pi/4) + f(\pi/2)\right) = S$$

2. Consider the following integral

$$\int_0^4 e^x \, dx$$

(a) (3 pts.) Implement Simpson's method using Matlab to solve this integral. Your code should print (i) number of data points used, (ii) step size, (iii) value of the integral, and (iv) exact error[1]. Print your code and include it in your solution. Name your code as `simpson.m` and submit soft copy as an e-mail attachment to `caner@uga.edu`.

SOLUTION.

```
function simpson

n=32

a=0;
b=4;

h = (b-a)/n;
sum = h*(exp(a)+exp(b))/3;

for i=1:2:(n-1)
  sum = sum + 4*h*exp(a+i*h)/3;
end

for j=2:2:(n-2)
  sum = sum + 2*h*exp(a+j*h)/3;
end

sum
err=abs(exp(4)-sum-1)
```

(b) (3 pts.) Find the order of Simpson's method by running your code using various step-sizes. Include the output of your runs, and show clearly the operations you did to estimate the order.

SOLUTION. Here is the output.

```
>> hw13_simpson
n =
     4
```

[1]Compute the error using the exact value of the integral.

```
sum =
   53.8638
err =
    0.2657
>> hw13_simpson
n =
     8
sum =
   53.6162
err =
    0.0181
>> hw13_simpson
n =
    16
sum =
   53.5993
err =
    0.0012
>> hw13_simpson
n =
    32
sum =
   53.5982
err =
   7.2562e-05
>>
>> 0.2657 / 0.0181
ans =
   14.6796
>> 0.0181 / 0.0012
ans =
   15.0833
>> 0.0012 / 7.2562e-05
ans =
   16.5376
>>
```

The ratio of errors when step-size is halved is approximately 16 fold. Therefore Simpson's rule must be a forth order method.

3. (3 pts.) Find the quadrature formula

$$\int_{-1}^{1} f(x)\,dx \approx c\sum_{i=0}^{2} f(x_i)$$

that is exact for all quadratic polynomials.

SOLUTION. There are four unknowns on the right hand side: c, x_1, x_2 and x_3. And there are three equations. for $f(x) = 1, x, x^2$. We have an additional degree of freedom. Let's see if we can find c

and α such that

$$x_1 = -\alpha,\, x_2 = 0,\, x_3 = \alpha$$

For $f(x) = 1, x, x^2$, we have:

$$\begin{aligned} 2 &= 3c \\ 0 &= 0 \\ \frac{2}{3} &= 2c\alpha^2 \end{aligned}$$

Using these equations, we get $c = \frac{2}{3}$ and $\alpha = \sqrt{\frac{1}{2}}$.

4. (3 pts.) Determine coefficients A_0, A_1, and A_2 that make the formula

$$\int_0^2 f(x)\,dx \approx A_0 f(0) + A_1 f(1) + A_2 f(2)$$

exact for all polynomials of degree 3.

SOLUTION. There are three unknowns (A_0, A_1, A_2) but four equations (for $f(x) = 1, x, x^2, x^3$). We might not be able to find a solution. Let's see:

$$\begin{aligned} 2 &= A_0 + A_1 + A_2 \\ 2 &= A_1 + 2A_2 \\ \frac{8}{3} &= A_1 + 4A_2 \\ 4 &= A_1 + 8A_2 \end{aligned}$$

Difference of third and second equations give us $A_2 = \frac{1}{3}$. Using second equation, we get $A_1 = \frac{4}{3}$. We get $A_0 = \frac{1}{3}$ from first equation. Note that we did not use the last equation so far. However, these values satisfy the last equation as well. Therefore we have a solution. Note that this is nothing but Simpson's rule.