1. (7 pts.) Write down the Taylor series for natural logarithm function about 1. Evaluate $ln(1.1)$ by summing the first three non-zero terms in the Taylor series. Compare your result with the actual value of $ln(1.1) = 0.095310179804$. How many accurate digits did you get?

$$\ln(1+x) = \underbrace{\ln(1)}_{=0} + x\frac{1}{1} + \frac{x^2}{2}\left(-\frac{1}{1^2}\right) + \frac{x^3}{6}\left(\frac{2}{1^3}\right)$$

$\underline{x = 0.1}$

$$\ln(1.1) = 0.1 + (-0.005) + 0.000333\ldots$$

$$= 0.0\underline{9}\,\underline{5}\,\underline{3}33\ldots$$

3 accurate digits.

2. (7 pts.) Prove that the following formula is a valid approximation for the second derivative (using Taylor's expansion). Find the order of this approximation.

$$f''(x) \approx \frac{f(x-h) - 2f(x) + f(x+h)}{\cancel{2h}\ h^2}$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f''''(x)$$

$$-2f(x) = -2f(x)$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f''''(x)$$

$+$

$$f(x-h) - 2f(x) + f(x+h) = h^2 f''(x) + \frac{h^4}{12}f''''(x) + \cdots$$

$$\frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = f''(x) + \frac{h^2}{12}f^{(4)}(x) + \cdots$$

This is a second order approximation.

3. (7 pts.) Suppose that you are given the following table:

x	0.0	0.25	0.5	0.75	1.0
$f(x)$	1	1.2	1.6	2	3

Compute your estimate of

$$\int_0^1 f(x)\,dx$$

using Simpson's rule. ~~(most accurate)~~ (use all ~~information given~~ five data points)

$$h = \frac{1.0}{4} = \frac{1}{4}$$

$$\frac{1}{4}\cdot\frac{1}{3}\left(1 + 4\times 1.2 + 2\times 1.6 + 4\times 2 + 3\right)$$

$$= \frac{1}{4}\frac{1}{3}\left(1 + \underbrace{4.8 + 3.2}_{=8} + 8 + 3\right)$$

$$= \frac{1}{4}\frac{1}{3}(20) = \frac{\cancel{20}^{5}}{3\cdot\cancel{4}} = \frac{5}{3}$$

4. (7 pts.) Suppose that you are given a smaller table for the same function in Problem 3:

x	0.0	0.5	1
$f(x)$	1	1.6	3

Compute your best estimate of

$$\int_0^1 f(x)\,dx$$

using

(a) ~~(3 pts.)~~ Using Trapezoid rule with 1 trapezoid.

(b) ~~(3 pts.)~~ Using Trapezoid rule with 2 trapezoids.

(c) ~~(3 pts.)~~ Using Richardson's extrapolation combining the results in (a) and (b).

(a) $1 \cdot \left(\frac{1+3}{2}\right) = 2$

(b) $\frac{1}{2} \cdot \left(\frac{1+1.6}{2}\right) + \frac{1}{2}\left(\frac{1.6+3}{2}\right) = \frac{1.3 + 2.3}{2} = 1.8$

$$\frac{4 \cdot 1.8 - 2}{3} = \frac{7.2 - 2}{3} = \frac{5.2}{3}$$

5. (7 pts.) Find A, B and C so that the numerical integration rule of the form

$$\int_{-1}^{1} x f(x)\, dx \approx A f(-1) + B f(0) + C f(1)$$

is exact for all $f(x)$ polynomials of maximum degree m. What is m?

$\underline{f(x) = 1}$; $\underline{f(x) = x}$; $\underline{f(x) = x^2}$:

$$\int_{-1}^{1} x \cdot 1\, dx = 0 = A + B + C$$

$$\int_{-1}^{1} x \cdot x\, dx = \frac{2}{3} = -A + C$$

$$\int_{-1}^{1} x\, x^2\, dx = 0 = A + C \Rightarrow A = -C$$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad \frac{2}{3} = 2C, \qquad \underbrace{C = \frac{1}{3}, \quad A = -\frac{1}{3}}_{B = 0.}$$

Then $\int_{-1}^{1} x f(x)\, dx \approx -\frac{1}{3} f(-1) + \frac{1}{3} f(1)$

Just checking to see if it is valid for $f(x) = x^3$ as well:

$$\int_{-1}^{1} x \cdot x^3\, dx = \frac{1}{5} \neq -A + C = \frac{2}{3}$$

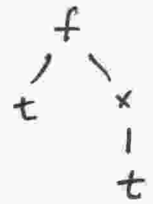
Therefore $\underline{m = 2}$

6. (7 pts.) Find $x(0.1)$ by solving the differential equation

$$\begin{cases} x' = -tx^2 \\ x(0) = 2 \end{cases}$$

with one step of the Taylor-series method of order 2.

$$x(t+h) \approx x(t) + h\underbrace{x'(t)}_{\substack{= f(t,x) \\ = -tx^2}} + \frac{h^2}{2}\underbrace{x''(t)}_{\substack{= \frac{d}{dt} f(t,x) \\ = -x^2 - 2tx(-tx^2)}}$$



$$x(t+h) \approx x(t) - htx^2 + \frac{h^2}{2}(2t^2x^3 - x^2)$$

$t = 0$
$h = 0.1$
$x = 2$

$$x(0.1) \approx 2 - 0.1 \cdot 0 + \frac{0.01}{2}(0 - 4)$$

$$\approx 2 + 0.01(-2)$$

$$\approx 1.98$$