Discrete Time Models

Lecture 1

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January 13, 2008

Model Classes

State $\longrightarrow$ Change $\longrightarrow$ New State

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- Difference Equations (discrete time)

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  - Discrete in time and space

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$$r = 1 - d + b \ : \ \text{Net growth rate}$$

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Question: When will the bacteria be equal to 10^9?

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Question: When will the bacteria be equal to 10^9?

$$2^n = 10^9 = (10^3)^3 \approx (2^{10})^3 = 2^{30}$$

Answer: $n = 30$ time units

Death

Birth rate (b) = 0

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General Solution:

$$N_t = (1 - d + b)^t N_0 = r^t\, N_0$$

Outcomes:

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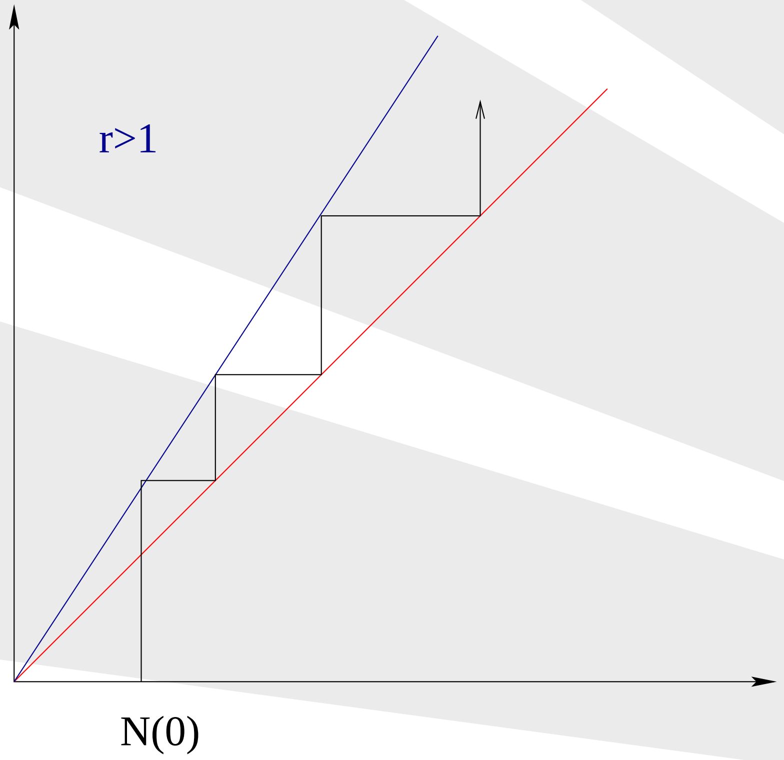
Outcomes: Exponential growth, exponential decay, constant solution.

Cobwebs

Net Growth rate ($r > 1$)

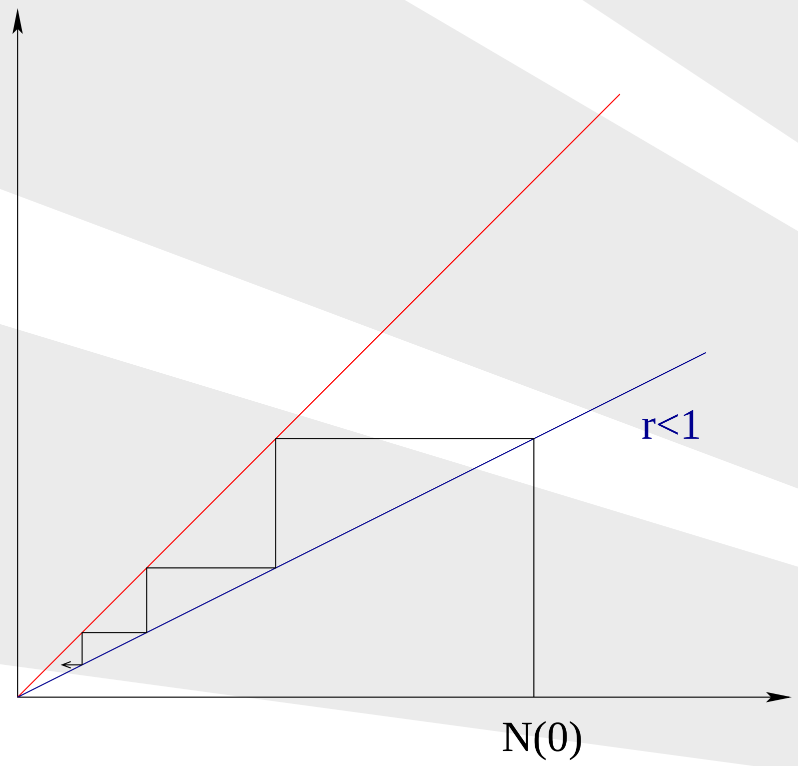
Exponential Growth



Cobwebs

Net Growth rate ($r < 1$)

Exponential Decay



Variable growth rate

Constant growth rate is not realistic.
Next Step: Growth rate depends on population

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$$
\begin{aligned}
r &= r(N) \\
r(N) &= c\left(1 - \frac{N}{K}\right)
\end{aligned}
$$

K : Carrying capacity

Variable growth rate

Constant growth rate is not realistic.
Next Step: Growth rate depends on population

$$\begin{aligned} r &= r(N) \\ r(N) &= c\left(1 - \frac{N}{K}\right) \end{aligned}$$

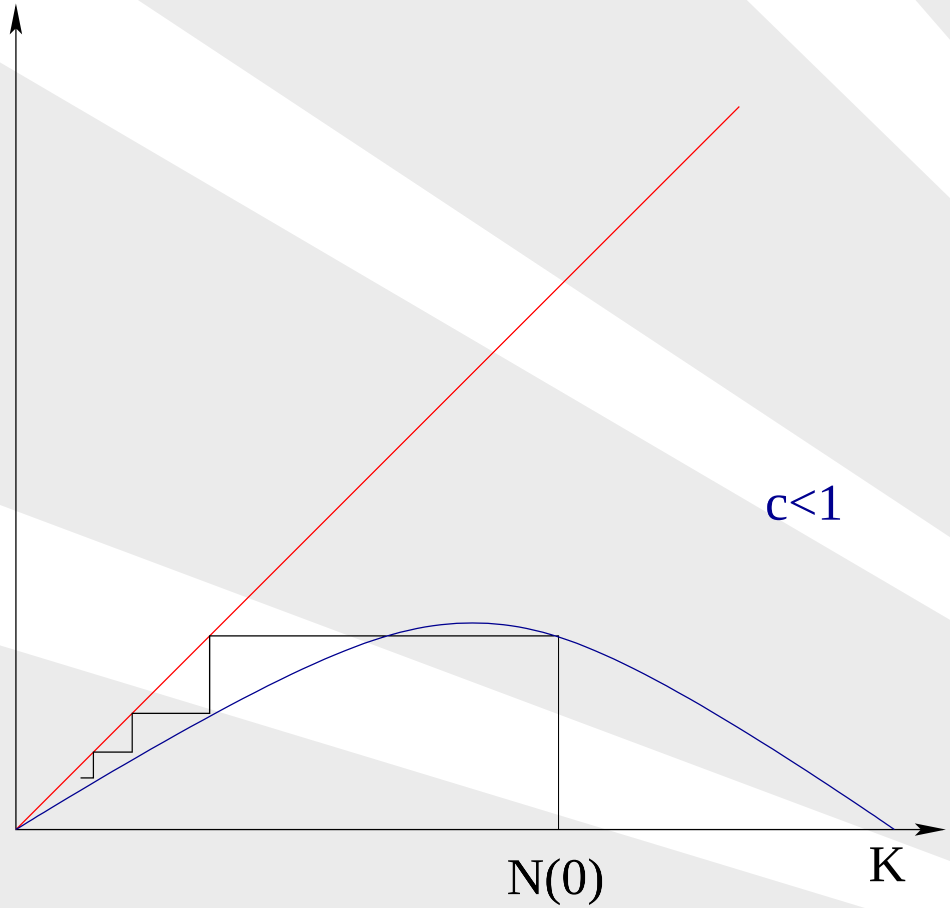
K : Carrying capacity
Let

$$n_t = \frac{N_t}{K}$$

Then

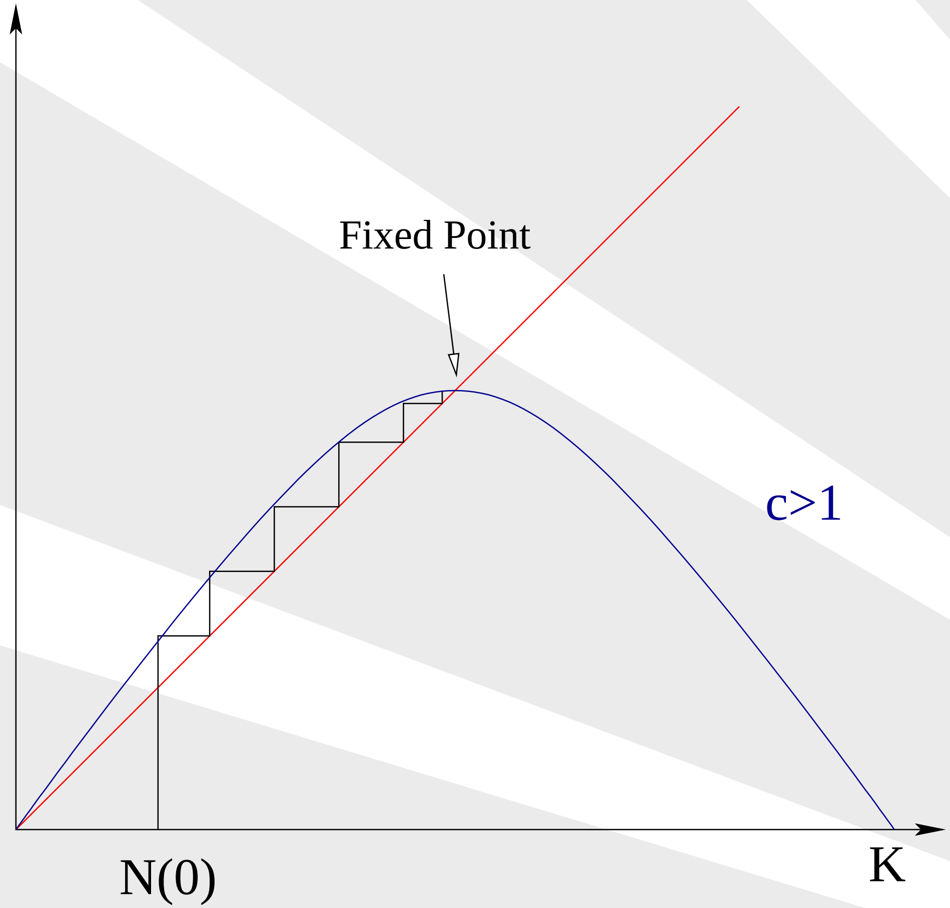
$$n_{t+1} = c\,(1 - n_t)\,n_t$$

Cobweb



$$n_{t+1} = c\left(1 - n_t\right) n_t \quad c < 1$$

Cobweb



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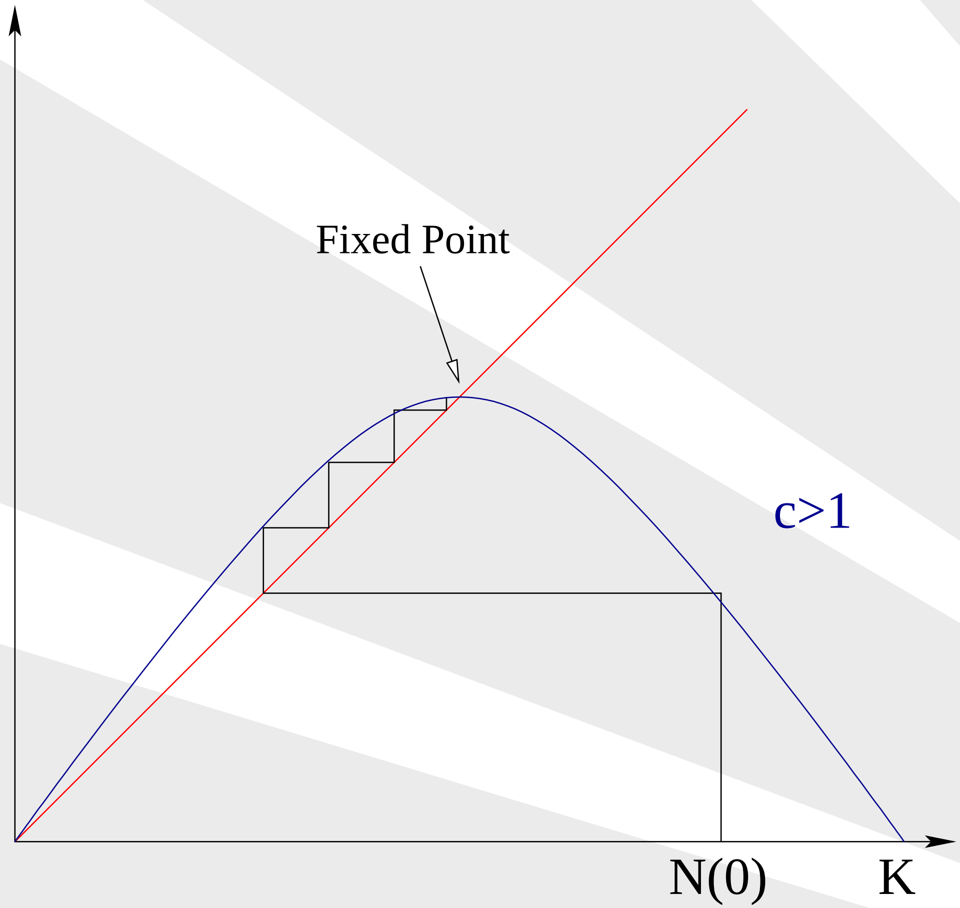

$$n_{t+1} = c\,(1 - n_t)\,n_t \quad c > 1$$

Discrete time population models

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Example:

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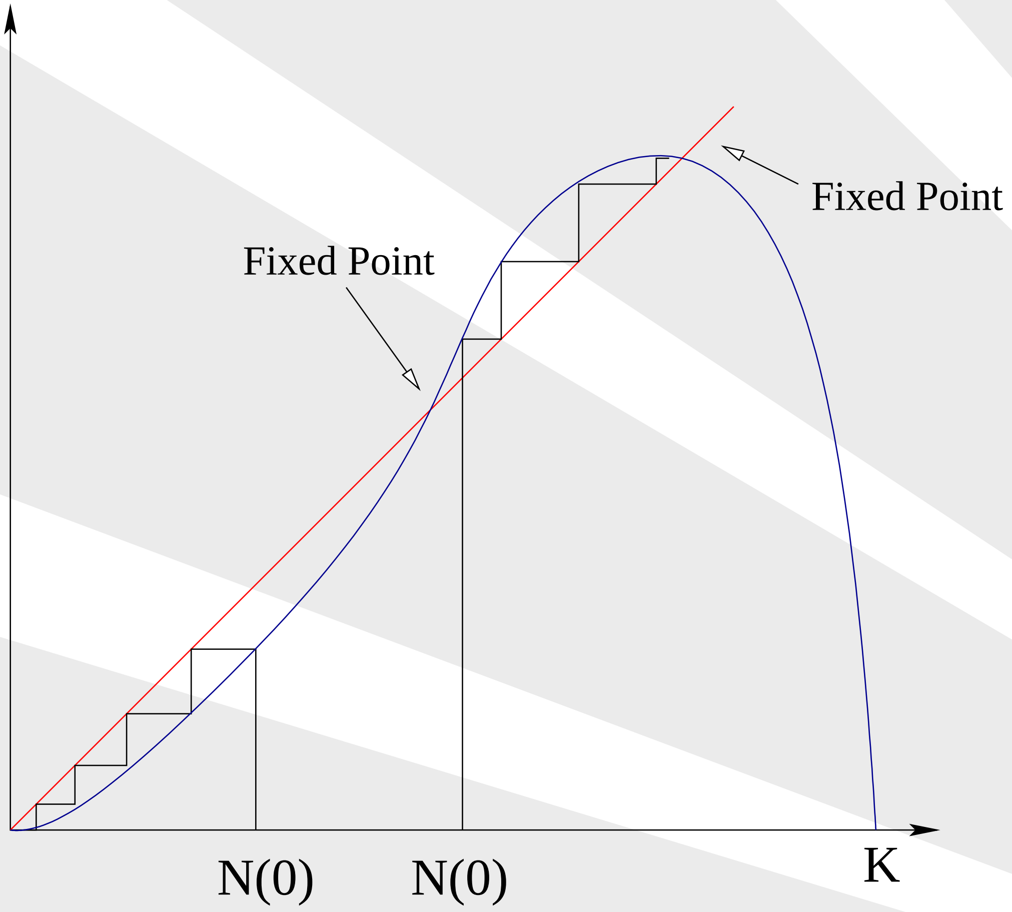
Rescale by K:

$$n_{t+1} = c\,(1 - n_t)\, n_t^2, \quad n = \frac{N}{K}$$

Cobweb



$$n_{t+1} = c\,(1 - n_t)\,n_t^2$$

From Discrete to Continuous

$$N_{t+1} = (1 - d + b)\, N_t(t)$$

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General case?

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Discrete time fixed points: $N_{t+1} = N_t = N^*$

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Discrete time fixed points: $N_{t+1} = N_t = N^*$
Continuous time fixed points: $f(N^*) = 0$

Long term behavior

Fixed growth rate model:

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With carrying capacity:

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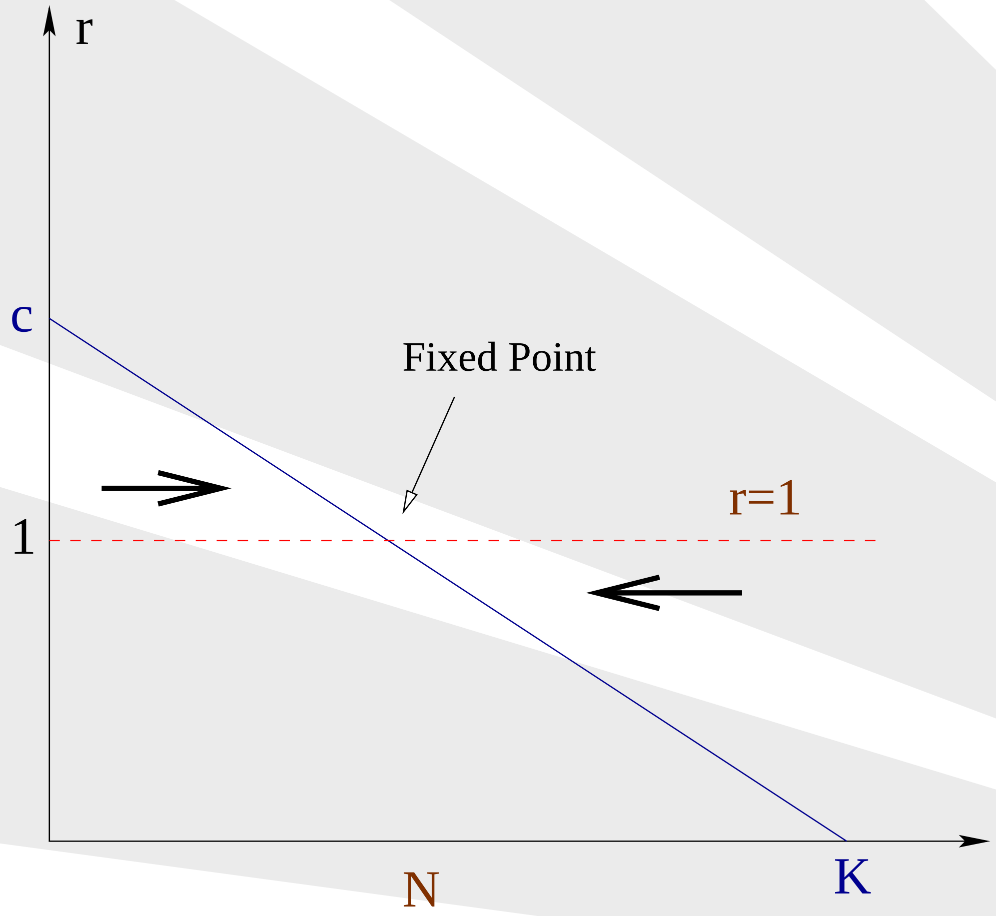
Is the fixed point $N^* = K$ stable?

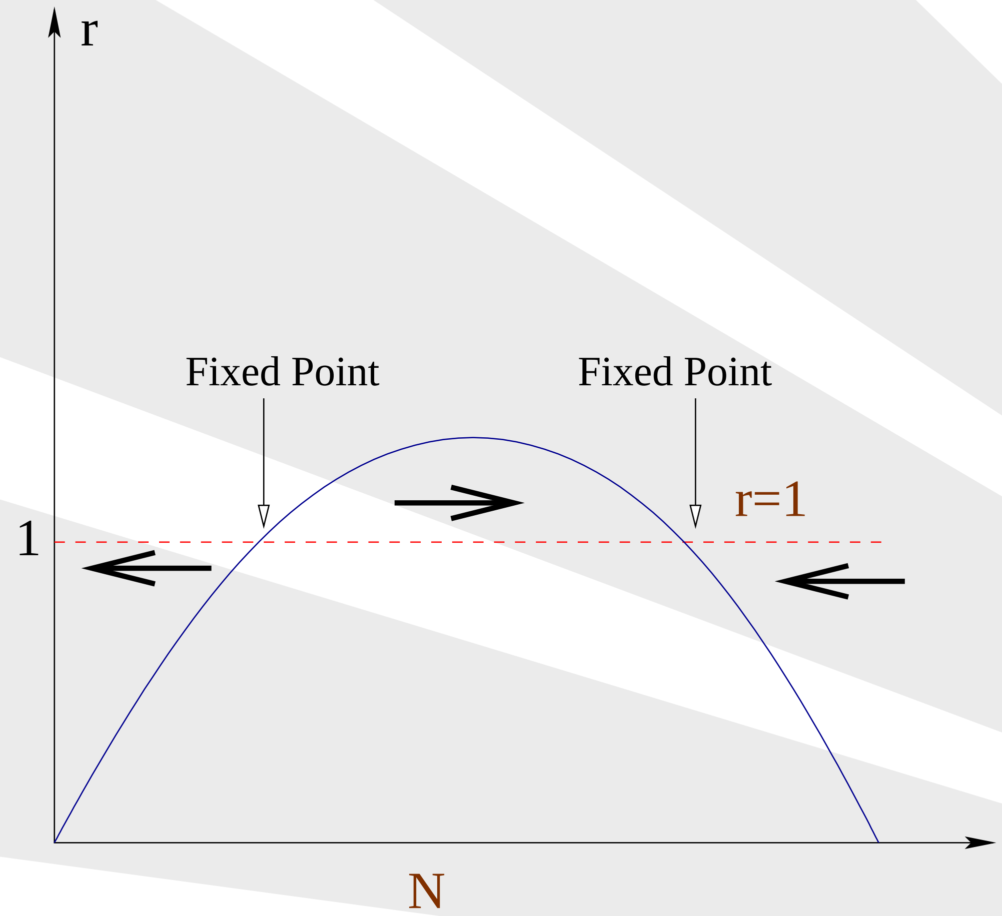
Stability

eg. $r(N) = c\,(1 - \frac{N}{K})$



Stability

eg. $r(n) = c\,n\,(1-n)$



Stability analysis fo fixed points

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x^* is a stable fixed point if $\beta < 1$