Stability of multi-dimensional discrete-time models

Lecture 4

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Stability analysis (1d)

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x^* is a stable fixed point if $|\beta| < 1$

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Combining above equations, we get:

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Taylor's explansion

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where $\epsilon = \begin{bmatrix} h \\ k \end{bmatrix}$ and $\nabla g = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} g_x \\ g_y \end{bmatrix}$.

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$$g(x+h, y+k) = g(x,y) + h\, g_x(x,y) + k\, g_y(x,y) + \cdots$$

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$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}$$

Eigen values

Theorem 2.2: The fixed points (x^*, y^*) are stable only if the absolute value of (both) eigen-values of

$$J = \begin{bmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{bmatrix}$$

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Jury (96) conditions:

$$|\mathrm{tr} J| < 1 + \det J < 2$$

Why?

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$$\lambda^2 - \mathrm{tr}J\,\lambda + \det J = 0$$