Phase Plane Analysis

Lecture 7

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General 2-d models

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  - Identify the stable/unstable manifolds

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  - Establish the nature of the fixed points
  - Identify the stable/unstable manifolds
- Draw nullclines

Phase plane analysis

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at the fixed points, and consider its eigen values.

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Theorem: Linear stability implies non-linear stability if none of the eigen values have zero real parts.

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Easy rule: Eigen values of a 2x2 matrix A have negative real parts if and only if the determinant of A is positive and the trace of A is negative.

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Note that the eigen values are:

$$\lambda_{1,2} = \frac{\mathrm{tr} A}{2} \pm \frac{1}{2}\sqrt{(\mathrm{tr} A)^2 - 4 \det A}$$

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- ◊ Spirals or vortices can be unstable or stable but have two complex eigenvalues with both positive or both negative real parts and occur when $D < 0$. Solutions spiral into or out of these fixed points.
- ◊ Saddles consist of one positive and one negative eigenvalue. They are unstable and occur when $D < 0$.

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- ◊ Consider the positive eigenvalue. Associated with this is the corresponding eigenvector. There exists a pair of solutions which leave the fixed point tangent to this eigenvector, and these are called the unstable manifold.
- ◊ Consider the negative eigenvalue. Similarly, there is a unique pair of solutions which come into the fixed point tangent to the eigenvector associated with the negative eigen value, called the stable manifold.

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- $y - nullcline$ is the set of points where $g(x, y) = 0$.
- Intersections of nullclines are places where $f(x, y) = 0 = g(x, y)$. These are fixed points!