Infectios Diseases

Lecture 8

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Outline

- ❖ Questions
- ❖ Classes of Individuals
- ❖ Possible Scenarios
- ❖ Simple Disease Epidemics (SI)
- ❖ Model (SI)
- ❖ Long term behavior (SI)
- ❖ Simple Disease Epidemics (SIS)
- ❖ SIR Epidemics
- ❖ Simplified SIR model
- ❖ Stability (SIR)
- ❖ SIR Endemics
- ❖ Simplified SIR Endemics

Questions

Classes of Individuals

Possible Scenarios

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Model (SI)

Long term behavior (SI)

Simple Disease Epidemics (SIS)

SIR Epidemics

Simplified SIR model

Stability (SIR)

SIR Endemics

Simplified SIR Endemics

Questions

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Questions we would like to answer:

- Will there be an epidemic?
- If so, how many individuals will be affected?
- If the disease is endemic, what is the prevalence (total number of cases at a given time) of the infection?
- Can the disease be eradicated or controlled?

Translation into mathematical questions:

- What is the long term behavior of the system?
- Can we modify the long term behavior by perturbing available parameters?
- Bifurcation Analysis

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- **Susceptible**: Can be infected
- **Latent (exposed)**: Infected, but not yet infectious
- **Infectious**: Infects others, but symptoms need not appear yet.
- **Removed**: No longer infectious, immune, isolated or dead.
- **Carrier**: Infectious, but do not show symptoms.
- **Incubation period**: The period before symptoms appear.

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- Susceptible $\rightarrow$ Latent $\rightarrow$ Infectious $\rightarrow$ Removed
- Susceptible $\rightarrow$ Latent $\rightarrow$ Carrier
- Long term behavior:
  - **Epidemic** diseases: prevalent only at particular times
  - **Endemic** diseases: habitually prevalent

Assumption: Population is homogeneous in mixing (space) and age (type).

Simple Disease Epidemics (SI)

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Assumptions:

- Disease is contagious
  - Spread by contact between S and I
- No incubation period
  - No delay ($S \to I$)
- Population is closed
  - $S + I$ remains constant.
- No recovery
  - Only two variables: S, I

Model (SI)

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$$\dot{S} = -f(S, I), \qquad \dot{I} = f(S, I)$$

$f(S, I)$: Incidence of the disease, the rate at which infections occur.

$f(S, I)$ is an increasing function for both S and I.

Assumption:

$$f(S, I) = \beta I S$$

β : Force of infection coefficient.

Similar to 'law of mass action'. Note that f is linear in terms of both S and I. Is it reasonable?

Long term behavior (SI)

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$$\dot{S} = -\beta S I, \qquad \dot{I} = \beta S I$$

Using conservation

$$m = S + I$$

we get a simpler model:

$$\dot{I} = \beta(m - I)I$$

What is the long term behavior?
How can we modify it for regular disease?

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Additional assumptions:

- Recovery is possible ($S \to I \to S$)
- No immunization (Recovered individuals become Susceptible)

$$\dot{S} = -f(S,I) + g(I), \qquad \dot{I} = f(S,I) - g(I)$$

Assuming linear recovery:

$$\dot{S} = -\beta SI + \gamma I, \qquad \dot{I} = \beta SI - \gamma I$$

γ : Recovery rate

What is the long term behavior?
How can we modify it for regular disease?

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Recovery (Immune) or death is possible:

$$S \to I \to R$$

Population is conserved:

$$m = S + I + R$$

Model:

$$\begin{aligned} \dot{S} &= -\beta I S \\ \dot{I} &= \beta I S - \gamma I \\ \dot{R} &= \gamma I \end{aligned}$$

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$$\begin{aligned} \dot{S} &= -\beta I S \\ \dot{I} &= I(\beta S - \gamma) \end{aligned}$$

Fixed points at $I = 0,\ S = S$.
Stability at this fixed point (no Infectious):

$$J(S,0) = \begin{pmatrix} 0 & -\beta S \\ 0 & \beta S - \gamma \end{pmatrix}$$

Eigenvalues: $0,\ \beta S - \lambda$.

Stability (SIR)

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$$J(S,0) = \begin{pmatrix} 0 & -\beta S \\ 0 & \beta S - \gamma \end{pmatrix}$$

Eigenvalues: $0,\ \beta S - \gamma$.

- Fixed point is stable (not asymptotically) along the eigen-vector corresponding to the zero eigenvalue. $(I = 0)$
- Fixed point is aymptotically stable or unstable (depending on the sign of $\beta S - \lambda$) along the eigen vector corresponding to the second eigen value.

Disease free state is not stable when $\beta S - \gamma > 0$ so an epidemic can occur.

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- Disease habitually prevalent
- Disease duration is longer than host life

Population is conserved:

$$m = S + I + R$$

Model:

$$\begin{aligned} \dot{S} &= \alpha m - \beta I S - \alpha S \\ \dot{I} &= \beta I S - \gamma I - \alpha I \\ \dot{R} &= \gamma I - \alpha R \end{aligned}$$

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$$\begin{aligned} \dot{S} &= \alpha(m - S) - \beta I S \\ \dot{I} &= I(\beta S - \gamma - \alpha) \end{aligned}$$

There exists a non-trivial fixed point at $S = \frac{\gamma+\alpha}{\beta}$ and the corresponding I value, obtained from $\dot{S} = 0$.

Stability can be analyzed using XPP.