Stochastoic Models

Lecture 11

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What is a stochastic model?

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$$x(t) \quad \text{vs} \quad P(x = a|t)$$

- While the outcome may be uncertain, probability of the outcome may very well be predetermined.

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$$ODE \longrightarrow SDE \longrightarrow PDE$$

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- The state is uncertain only because the transition is uncertain.

Adding «noise»

$$S + E \leftrightarrows C \rightarrow P + E$$

```
% ODE model of S(t) using Michaelis-Menten kinetics
% Solution using Euler's method
km = 3;       % Parameters
vm = 5.6;
s(1) = 25;    % Initial consitions
e(1) = .02;
dt=.001;      % step-size
for i=1:1000
   s(i+1)=s(i)-dt*vm*s(i)/(km+s(i));
   t(i+1)=t(i)+dt;
end
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for i=1:1000
   s(i+1)=s(i)-dt*vm*s(i)/(km+s(i))+0.2*randn;
   t(i+1)=t(i)+dt;
end
```

Markov Process

- Memoryless, discrete time stochastic process

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eg. Forrest dynamics

When a red oak tree dies, it is equally likely to be replaced by a hickory or red oak,

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When a red oak tree dies, it is equally likely to be replaced by a hickory or red oak, and that when a hickory tree dies, it has probability 0.74 of being replaced by a red oak, and 0.26 of being replaced by hickory.

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o_t : ratio of red oak trees at time t

h_t : ratio of hickory trees at time t

$$\begin{bmatrix} o_{t+1} \\ h_{t+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.74 \\ 0.5 & 0.26 \end{bmatrix} \begin{bmatrix} o_t \\ h_t \end{bmatrix}$$

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What happens in the long term?

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What happens in the long term? $(u^*, v^*) = (0.597, 0.403)$

Markov Chain

- Probability vector

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- Transition matrix

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Note that, if $u_t \to u^*$, then $Pu^* = u^*$.

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Note that, if $u_t \rightarrow u^*$, then $Pu^* = u^*$. So u^* is the eigen vector corresponding to the eigen value $\lambda = 1$.