Introduction to Stochastic Algorithms: Basic Stochastic Algorithm & Gillespie's Stochastic Algorithm

Lecture 12

Caner Kazancı

Mathematical Biology
University of Georgia
March 19-24, 2009

Deterministic Approach

Initial value problem based on the reaction:

$$\text{IVP} \begin{cases} \dot{x} = f(x) = V\,a(x) \\ x(t_0) = x_0 \end{cases}$$

Here $x(t)$ is a unique deterministic path based on the initial conditions and mass action kinetics.

Probabilistic Approach

Probability that the j^{th} reaction will occur once in the next infinitesimal time interval $[t, t + h]$ is approximately equal to $a_j(x)\, h$.

Probabilistic Approach

Probability that the j^{th} reaction will occur once in the next infinitesimal time interval $[t, t+h]$ is approximately equal to $a_j(x)\,h$.

A random number r for each reaction is chosen

Probabilistic Approach

Probability that the j^{th} reaction will occur once in the next infinitesimal time interval $[t, t+h]$ is approximately equal to $a_j(x)\,h$.

A random number r for each reaction is chosen, and if $a_j(x)\,h > r$ holds,

Probabilistic Approach

Probability that the j^{th} reaction will occur once in the next infinitesimal time interval $[t, t+h]$ is approximately equal to $a_j(x)\,h$.

A random number r for each reaction is chosen, and if $a_j(x)\,h > r$ holds, then we assume the reaction occurs during $[t, t+h]$

Probabilistic Approach

Probability that the j^{th} reaction will occur once in the next infinitesimal time interval $[t, t+h]$ is approximately equal to $a_j(x)\,h$.

A random number r for each reaction is chosen, and if $a_j(x)\,h > r$ holds, then we assume the reaction occurs during $[t, t+h]$ and the state of the system are updated according to the state shift vector v_j corresponding to that reaction

Probabilistic Approach

Probability that the j^{th} reaction will occur once in the next infinitesimal time interval $[t, t+h]$ is approximately equal to $a_j(x)\,h$.

A random number r for each reaction is chosen, and if $a_j(x)\,h > r$ holds, then we assume the reaction occurs during $[t, t+h]$ and the state of the system are updated according to the state shift vector v_j corresponding to that reaction as follows

$$x(t+h) = x(t) + v_j$$

where x is the actual number of molecules rather than molar concentrations.

Basic Stochastic Algorithm

```
input n =#iterations, r =#reactions, h =step size (dt)
input x(0) =initial conditions, v =state shift matrix
t ← 0
for i = 1 to n
.    for j = 1 to r
.    .    p ←uniform random number in [0,1]
.    .    If p < a(x)h then x(t + h) ← x(t) + v_j
.    end
.    t ← t + h
end
```

Basic Stochastic Algorithm

```
input n =#iterations, r =#reactions, h =step size (dt)
input x(0) =initial conditions, v =state shift matrix
t ← 0
for i = 1 to n
.    for j = 1 to r
.    .    p ←uniform random number in [0,1]
.    .    If p < a(x)h then x(t + h) ← x(t) + v_j
.    end
.    t ← t + h
end
```

Note that the time step h should be small enough so that $a_j(x)\, h < 1,\ \forall j \in \{1, \ldots, r\}$ holds throughout the simulation.

Gillespie's Stochastic Method

- Find the expected time when the next reaction will occur.
- Find which reaction out of all reactions in the network will occur next.

Gillespie's Stochastic Method

- Find the expected time when the next reaction will occur.
- Find which reaction out of all reactions in the network will occur next.

To find the expected time when the next reaction will occur, we compute the probability $\mathcal{P}(T|x, t)$ that the next reaction will occur immediately after T seconds.

Gillespie's Stochastic Method

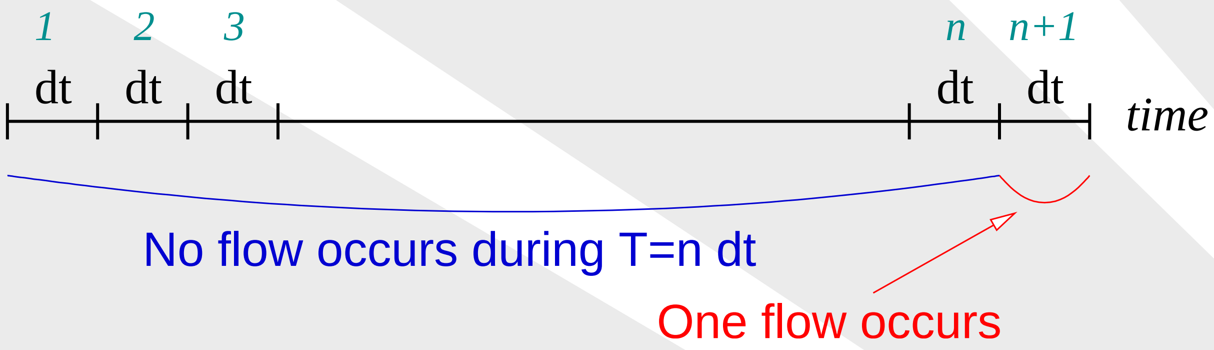
- Find the expected time when the next reaction will occur.
- Find which reaction out of all reactions in the network will occur next.

To find the expected time when the next reaction will occur, we compute the probability $\mathcal{P}(T|x,t)$ that the next reaction will occur immediately after T seconds.

Note that the probability that any reaction will happen during $[t, t+h]$ is $s(x)\,h$ where

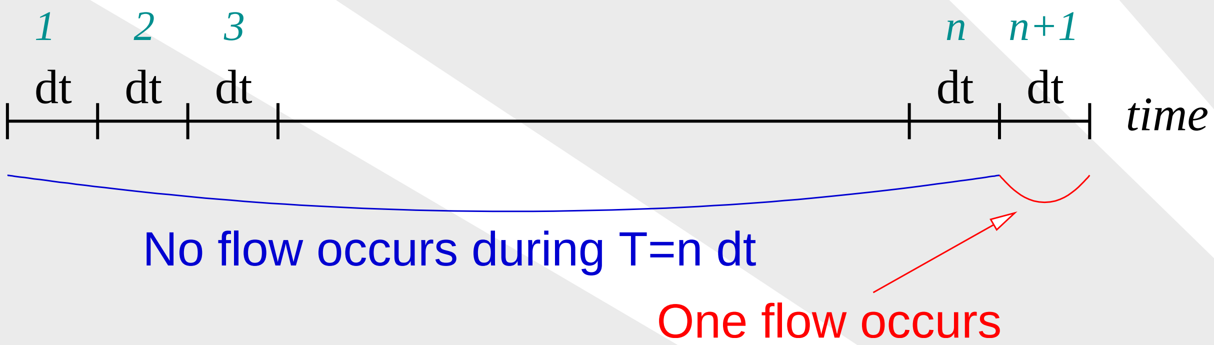
$$s(x) = \sum_{j=1}^{r} a_j(x)$$

Gillespie's Stochastic Method



(h=dt)

Gillespie's Stochastic Method

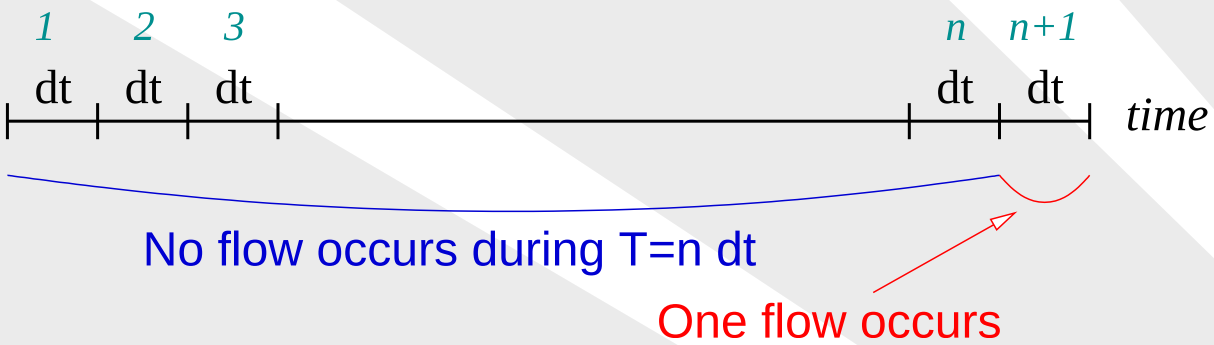


(h=dt)

Then the probability that no reaction will happen during $[t, t+T]$ and that a single reaction will occur during $[t+T, t+T+h]$ is

$$\mathcal{P}(T|x,t) = (1 - s\,h)^n s\,h \quad \text{where} \quad T = n\,h$$

Gillespie's Stochastic Method



(h=dt)

Then the probability that no reaction will happen during $[t, t+T]$ and that a single reaction will occur during $[t+T, t+T+h]$ is

$$\mathcal{P}(T|x,t) = (1 - s\,h)^n s\,h \quad \text{where} \quad T = n\,h$$

Let $h \rightarrow 0$ and we get

$$\mathcal{P}(T|x,t) = s\,e^{-Ts}\,h$$

Gillespie's Stochastic Method

We define the next reaction probability distribution function based on this equation as follows:

$$\mathrm{P}(T) = \begin{cases} s\, e^{-Ts} & ; \quad T \geq 0 \\ 0 & ; \quad T < 0 \end{cases}$$

Gillespie's Stochastic Method

We define the next reaction probability distribution function based on this equation as follows:

$$\mathrm{P}(T) = \begin{cases} s\, e^{-Ts} & ; \quad T \geq 0 \\ 0 & ; \quad T < 0 \end{cases}$$

At each iteration step, we would like to choose the step length T according to this probability distribution.

Gillespie's Stochastic Method

We define the next reaction probability distribution function based on this equation as follows:

$$\mathrm{P}(T) = \begin{cases} s\, e^{-Ts} & ; \quad T \geq 0 \\ 0 & ; \quad T < 0 \end{cases}$$

At each iteration step, we would like to choose the step length T according to this probability distribution. If r is a uniform random variable on $[0,1]$ then the following choice of T gives the desired random time interval:

$$T = -\frac{1}{s} \ln r$$

Gillespie's Algorithm

```
input $n$ =#iterations, $r$ =#reactions
input $x(0)$ =initial conditions, $v$ =state shift matrix
$t \leftarrow 0$
for $i = 1$ to $n$
.    $s \leftarrow \sum_{i=1}^{r} a_i(x)$
.    $p1$ =random number in [0,1]
.    $h \leftarrow -\log(p1)/s$
.    $t \leftarrow t + h$
.    $p2$ =random number in [0,1]
.    $ctr \leftarrow 0$
.    for $j = 1$ to $r$
.    .    if $ctr < p2 < ctr + a_j/s$ then $x(t+h) \leftarrow x(t) + v_j$
.    .    $ctr \leftarrow ctr + a_j/s$
.    end
end
```