Concrete Interpolation of Meromorphic Matrix Functions on Riemann Surfaces

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To our friend Harry Dym on the occasion of his 60^{th} birthday

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ABSTRACT. This work investigates concrete problems of interpolating matrix pole-zero data with multiple-valued meromorphic matrix functions on closed Riemann surfaces. In the case of genus $g > 1$, a condition sufficient for the existence of a solution having constant factor of automorphy is presented. Necessary and sufficient conditions are presented in the case where $g = 1$. A necessary and sufficient condition for single-valued matrix function interpolation in arbitrary genus is also established.

This paper deals with the problem of interpolating matrix pole-zero data by regular meromorphic matrix functions on a closed Riemann surface M of genus greater than zero. In classical formulations of such interpolation problems, the data is given as a matrix divisor. A matrix divisor Θ is a section the sheaf of germs of regular $r \times r$-meromorphic matrix functions on M modulo one-side equivalence by invertible analytic matrix functions. This notion of divisor was introduced by [14]. In general, there will exist a multiple-valued $r \times r$-meromorphic matrix function G interpolating the divisor in the sense that the matrix function germ determined by G belongs to the value of Θ at points of M. This last result is a consequence of the triviality of vector bundles on the universal cover $\rho : \widetilde{M} \to M$ of M. In fact, if $\mathcal{G}$ is the group of covering transformations for $\rho : \widetilde{M} \to M$, then the vector bundle on M determined by Θ corresponds to a holomorphic matrix factor of automorphy $\xi : \mathcal{G} \times \widetilde{M} \to GL(r, \mathbf{C})$. By definition this matrix factor of automorphy satisfies

$$\xi(ST, u) = \xi(S, Tu)\xi(T, u),\ \ S, T \in \mathcal{G},\ \ u \in \widetilde{M}.$$

There is an $r \times r$-meromorphic matrix function $G = G(u)$ on $\widetilde{M}$ satisfying

$$G(Tu) = \xi(T, u)G(u),\ \ (T, u) \in \mathcal{G} \times \widetilde{M}$$

such that the germs of G agree with the values of the divisor $\rho^*\Theta$ at points of $\widetilde{M}$. This abstract solution of the interpolation problem is a nice existence theorem; however, to understand the multiple-valued nature of the solution to the interpolation problem, one must determine the factor of automorphy ξ that is associated with the divisor data Θ. One result in this direction appears in a classic paper of Weil [14]. Weil gives finite dimensional necessary and sufficient conditions for the existence of a solution to the above interpolation problem that has constant factor of automorphy, i.e., associated to a representation of $\mathcal{G}$. The existence of such a solution is equivalent to the condition that the vector bundle associated with Θ is flat.

The goal of this paper is to study the abstract interpolation problem described above in a more concrete context. In many applications, it is important to present the interpolation data in a form that explicitly displays the matrix pole-zero data at the interpolation nodes. A linear algebra description of the pole-zero data of a meromorphic matrix function has recently been given using the concept of a *null-pole triple* as described in [4]. This triple will be described in greater detail below. For now, we are content with a indication of the nature of null-pole triples. Suppose that $F = F(z)$ is a regular $r \times r$-meromorphic matrix function defined in a neighborhood of $z = 0$ in $\mathbf{C}$. The (right) null-pole triple of F at $z = 0$ has the form

$$\Upsilon = ((B_\zeta, A_\zeta), (A_\pi, C_\pi), S).$$

In this triple, the pair of matrices (A_π, C_π), where A_π is $n_\pi \times n_\pi$ and C_π is $n_\pi \times r$, captures the pole behavior of F, in the sense that for some matrix $\widetilde{B}$ the difference

$$F(z) - \widetilde{B}(zI - A_\pi)^{-1}C_\pi$$

is analytic at $z = 0$ and the pair of matrices (B_ζ, A_ζ), where A_ζ is $n_\zeta \times n_\zeta$ and B_ζ is $r \times n_\zeta$, captures the zero behavior of F in the sense that for some matrix $\widetilde{C}$ the difference

$$F^{-1}(z) - B_\zeta(zI - A_\zeta)^{-1}\widetilde{C}$$

is analytic at $z = 0$. The $n_\pi \times n_\zeta$ matrix S satisfies $A_\pi S - SA_\zeta = C_\pi B_\zeta$ and is called the coupling matrix[9]. In a somewhat imprecise sense, the matrix S accounts for any pole-zero cancellation in $detF$ at $z = 0$. There are natural concepts of minimality and similarity for null-pole triples such that the similarity orbits of minimal null-pole triples are in a one-to-one correspondence with matrix divisors at $z = 0$. As a consequence of this last fact, the problem of interpolating matrix divisors is equivalent to the more concrete problem of interpolating null-pole triples. The concrete formulation of this interpolation problem comes with a price. Namely, the prescription of a matrix null-pole triple at a point on the Riemann surface must be done in specific local coordinates. (Below, we will offer a coordinate free method of describing null-pole triples.) If one exploits uniformization, then the dependence of the interpolation data on local coordinates is somewhat diminished (or at least hidden). Let $z_1, ..., z_K$

be points on M. We will specify interpolation data on M of the form

$$\mathcal{D}\colon \{(\Upsilon_1, z_1), \cdots, (\Upsilon_K, z_K)\},$$

where Υ_j is an admissible null-pole triple prescribed in local coordinates (s_j, V_j) at the point z_j, with $s_j(z_j) = 0$, $j = 1, \cdots, K$. One obvious question is the following: Given the data $\mathcal{D}$, find necessary and sufficient conditions in terms of the data $\mathcal{D}$ for the existence of a global meromorphic matrix function G such that near z_j , $G(p) = F_j(s_j(p))$, where Υ_j is a null-pole triple of the matrix function F_j at zero, $j = 1, \cdots, K$. We will give an answer to this question. See, e.g., Theorem 8. A related question is to find necessary and sufficient conditions involving the data $\mathcal{D}$ for the existence of a multiple-valued meromorphic matrix function G that solves the interpolation problem such that G has a constant matrix factor of automorphy. We will give a complete solution to this problem in the case of genus one (see, Theorem 1) and a sufficient condition for a solution to this problem in higher genus (see, Corollary 6). The genus one result will take advantage of Atiyah's [1] description of vector bundles over an elliptic curve . In the case where M is realized in the form $M = \mathbf{C}/\mathbf{Z} + \tau\mathbf{Z}$, where $\Im\tau > 0$, the results of Atiyah [1] imply that a flat bundle E over M admits a representation

$$E = \mathbb{L}_1 \otimes F_{h_1} + ... + \mathbb{L}_s \otimes F_{h_s}.$$

In this representation $\mathbb{L}_1, ..., \mathbb{L}_s$ are degree zero line bundles and F_h denotes a rank h flat bundle which corresponds to the representation $\xi_h : \mathbf{Z} + \tau\mathbf{Z} \rightarrow GL(h, \mathbf{C})$ given on the generators of $\mathbf{Z} + \tau\mathbf{Z}$ by

$$\xi_h(1) = I_h \ : \ \xi_h(\tau) = \begin{bmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Perhaps a more natural question from the point of view of the structure of vector bundles is which interpolation data sets give rise to flat unitary vector bundles, or more generally, to “stable” or “semistable” bundles (see [13] for definitions and a full account of these concepts); we will also discuss some results on this problem in Section 3.

We close the introduction by describing a sample of our results in the simplest case on a torus where all poles and zeros are first order and occur at separate points.

Suppose that M is the complex torus $M = \mathbf{C}/\mathbf{Z} + \tau\mathbf{Z}$, where $\Im\tau > 0$. Consider the interpolation data

$$\mathcal{D} : \{(\underline{b}_1, z_1), ..., (\underline{b}_N, z_N) : (\mathbf{c}_1, w_1), ..., (\mathbf{c}_N, w_N)\},$$

where, $\underline{b}_1, ..., \underline{b}_N$ are r-dimensional column vectors, $\mathbf{c}_1, ..., \mathbf{c}_N$ are r-dimensional row vectors and $z_1, \cdots, z_N; w_1, \cdots, w_N$ are $2N$ distinct points on M. We look for multiple valued $r \times r$-meromorphic matrix functions G on M such that the poles of entries are at most simple poles at the points $w_1, \ldots, w_N$, the only zeros of $\det G$ are simple zeros at $z_1, \ldots, z_N$ with $\underline{b}_j$ spanning the (right-)kernel of $G(z_j)$ $(j = 1, ..., N)$ and at w_i, G^{-1} is analytic with $\mathbf{c}_i$ spanning the left-kernel of $G^{-1}(w_i),\ i = 1, ..., N$. The data set $\mathcal{D}$ can be lifted to the data set $\rho^*\mathcal{D}$ on $\mathbf{C}$. Any single-valued $r \times r$ -meromorphic matrix function $G = G(u)$ interpolating $\rho^*\mathcal{D}$ is to be considered a multiple-valued solution to the interpolation problem with data set $\mathcal{D}$. We introduce the classical theta function with characteristics $\{\frac{1}{2}, \frac{1}{2}\}$

$$\theta_*(u) = \sum_{n\in\mathbb{Z}} \exp\left\{2\pi i[\frac{1}{2}(n+\frac{1}{2})\tau(n+\frac{1}{2}) + (n+\frac{1}{2})(u+\frac{1}{2})]\right\},$$

associated with the lattice $\mathbf{Z} + \boldsymbol{\tau}\mathbf{Z}$.

Theorem 1. *In order that there exist a multiple-valued $r \times r$-meromorphic matrix function $G = G(u)$ solving the interpolation problem with data $\mathcal{D}$ that has automorphic behavior*

$$G(u+1) = G(u),\ G(u+\tau) = CG(u+\tau),$$

where C is an invertible $r \times r$-matrix, it is necessary and sufficient that for some $\lambda \neq 0$ the matrix

$$\Gamma_0^\lambda = \left[\frac{\theta_*(w_i - z_j + \lambda)\mathbf{c}_i\underline{b}_j}{\theta_*(w_i - z_j)}\right]_{N\times N}$$

is invertible.

The proof of this theorem provides additional information. There is a natural vector bundle $E_\mathcal{D}$ associated with the interpolation data $\mathcal{D}$.

Corollary 2. *The zeros of the determinant of the matrix function Γ_0^λ correspond to the line bundles providing the decomposition $E_\mathcal{D} = \mathbb{L}_1 \otimes F_{h_1} + ... + \mathbb{L}_s \otimes F_{h_s}$. The bundle $E_\mathcal{D}$ is equivalent to a direct sum of line bundles if and only if the determinant of Γ_0^λ has precisely r zeros (counting multiplicity) in a fundamental domain for $\dot{M}$. The bundle $E_\mathcal{D}$ is equivalent to the rank r bundle F_r if and only in the determinant of Γ_0^λ doesn't vanish.*

The genus zero version of Theorem 1 (where the torus is replaced by the Riemann sphere and the only flat bundle is the trivial bundle) goes back to [9]; see [4] for a complete treatment. Indeed, in the case where the zeros $z_1, ..., z_N$ and poles $w_1, ..., w_N$ are in the complex plane the invertibility of the $N \times N-$matrix $\Gamma = \left[\frac{\mathbf{c}_i \underline{b}_j}{w_i - z_j}\right]$ is necessary and sufficient for the existence of a rational matrix function solving the corresponding interpolation problem on the Riemann sphere. In the sequel, for arbitrary genus $g > 0$, using a matrix analogous to Γ_0^λ, necessary and sufficient conditions will be given for the interpolation problem with data $\mathcal{D}$ to have a single valued solution. A result in this direction was given earlier in [2].

It develops that every flat vector bundle on a closed Riemann surface is equivalent to an interpolation vector bundle associated with data of the simple form $\mathcal{D}$[3]. Thus the result in Theorem 1 gives conditions for realizing any flat vector bundle on a Riemann surface of genus $g = 1$ through global meromorphic functions on $\mathbb{C}$ having a simple pole-zero structure.

1. Local Divisors and Local Null Pole Triples

Let M be a Riemann surface. The sheaf of germs of holomorphic (respectively, meromorphic) functions on M will be denoted by $\mathcal{O}$ (respectively, $\mathcal{M}$). The notation $\mathcal{O}_p$ (respectively, $\mathcal{M}_p$) will be used for the stalk at $p \in M$ in $\mathcal{O}$ (respectively, $\mathcal{M}$). Further, we will denote $\mathcal{O}\otimes\mathbf{C}^r$ (respectively, $\mathcal{M}\otimes\mathbf{C}^r$) by $\mathcal{O}^r$ (respectively, $\mathcal{M}^r$) and $\mathcal{O}^{r\times r}$ (respectively, $\mathcal{M}^{r\times r}$) will denote the $r \times r$-matrix analogues. When convenient, elements in $\mathbb{C}^r$(respectively, $\mathcal{O}^r$) will be considered as row vectors (respectively, row vector functions). The regular elements in $\mathcal{O}^{r\times r}$ (respectively, $\mathcal{M}^{r\times r}$) will be denoted by $\mathcal{GL}(r, \mathcal{O})$ (respectively, $\mathcal{GL}(r, \mathcal{M})$). We let $\mathcal{GL}(r, \mathcal{O})$ act on $\mathcal{GL}(r, \mathcal{M})$ on the left. An element Θ_p in $(\mathcal{GL}(r, \mathcal{M})/\mathcal{GL}(r, \mathcal{O}))_p$ is called a (rank r) *local matrix divisor* at p. A *matrix divisor* is a section Θ of $\mathcal{GL}(r, \mathcal{M})/\mathcal{GL}(r, \mathcal{O})$. See, [12] for a slightly different definition of matrix divisor. The set of values $p \in M$ (necessarily finite) where $\Theta_p \neq I$ will be called the support of Θ. The value Θ_p of a matrix divisor is a set of germs at p of the form $[HF]_p$, where F is a fixed regular $r \times r$-meromorphic matrix function defined in a neighborhood of p and H varies over invertible $r \times r$-analytic matrix functions defined in a neighborhood of p. Two divisors Θ and $\widetilde{\Theta}$ are said to be *linearly equivalent* in case there is a globally defined regular $r \times r$-meromorphic matrix K such that $\widetilde{\Theta} = \Theta K$. If F is a regular $r \times r$-meromorphic matrix function defined in a neighborhood of p, then we define the *null-pole subspace* associated with F at p as

$$\mathcal{O}_p^r[F]_p = \{[\mathbf{f}]_p : \mathbf{f} = \mathbf{g}F, \ \mathbf{g} \text{ is (a row vector) } \mathbf{C}^r\text{-valued and analytic at p}\}. \quad (1)$$

This defines stalks of a locally free sheaf which is dual to the standard bundle determined by Θ. Obviously, if $[F]_p$ and $[\widetilde{F}]_p$ belong to the divisor Θ_p, then $\mathcal{O}_p^r[F]_p = \mathcal{O}_p^r[\widetilde{F}]_p$. Conversely, this last equality implies $[F]_p$ and $[\widetilde{F}]_p$ belong to the same divisor at p. As a consequence, if Θ_p is a matrix divisor at p, it determines a *null-pole subspace* $\mathcal{S}(\Theta_p)$ given by (1) in $\mathcal{M}^r$.

It develops that local matrix divisors admit a more concrete description in terms of matrix pole-zero data. In fact, there is a natural correspondence between local matrix divisors and similarity orbits of local null-pole triples. We briefly present the concept of local null-pole triple. Suppose F is an $r \times r$-meromorphic matrix function defined in a neighborhood of $z = 0$ in the complex plane. The pole-zero structure of F at $z = 0$ can be encoded in a *minimal right null-pole triple*

$$\Upsilon = ((B_\zeta, A_\zeta), (A_\pi, C_\pi), S), \tag{2}$$

where (B_ζ, A_ζ) is a minimal right zero pair of F, (A_π, C_π) is a minimal left pole pair of F and the $n_\pi \times n_\zeta-$null-pole coupling matrix S satisfies

$$A_\pi S - S A_\zeta = C_\pi B_\zeta. \tag{3}$$

To say that (B_ζ, A_ζ) is a minimal right zero pair of F means that A_ζ is an $n_\zeta \times n_\zeta$-nilpotent matrix, B_ζ is an $r \times n_\zeta$-matrix with

$$\bigcap_{j=0}^{n_\zeta} \ker(B_\zeta A_\zeta^j) = \{\mathbf{0}\} \tag{4}$$

and there is an $n_\zeta \times n-$ matrix $\widetilde{C}$ such that

$$F^{-1}(z) - B_\zeta (zI - A_\zeta)^{-1} \widetilde{C}$$

is analytic at zero. Note that if (B_ζ, A_ζ) is a left zero pair for F, then for any invertible matrix U

$$(\ B_\zeta U, U^{-1} A_\zeta U) \tag{5}$$

is also a left zero pair for F . Moreover, any left zero pair for F will have the form (5) for some invertible matrix U. In a dual manner, to say that (A_π, C_π) is a right pole pair of F means that A_π is an $n_\pi \times n_\pi$-nilpotent matrix, C_π is an $n_\pi \times r$ -matrix with

$$\sum_{j=0}^{n_\pi} \mathrm{Im}(A_\pi^j C_\pi) = \mathbb{C}^{n_\pi} \tag{6}$$

and there is an $r \times n_\pi-$ matrix $\widetilde{B}$ such that

$$F(z) - \widetilde{B}(zI - A_\pi)^{-1} C_\pi$$

is analytic at zero. If (A_π, C_π) is a right-pole pair of F, then for any invertible V

$$(V^{-1}A_\pi V, V^{-1}C_\pi) \tag{7}$$

is a right pole pair of F. Any right pole pair for F will have the form (7) for some invertible matrix V.

The fact that S acts as a coupling operator means the following: Given an r-dimensional row vector function $\mathbf{h} = \mathbf{h}(z)$ analytic at zero, then one can write

$$(\mathbf{h}F)(z) = \mathbf{x}(zI - A_\pi)^{-1}C_\pi + \mathbf{k}(z) \tag{8}$$

where $\mathbf{x}$ is an n_π-dimensional row vector and $\mathbf{k} = \mathbf{k}(z)$ is analytic at zero. Moreover, every n_π-dimensional row vector $\mathbf{x}$ occurs in such a decomposition for an appropriate choice of $\mathbf{h}$. The coupling operator S satisfies

$$\mathbf{x}S = res_{z=0}\left[\mathbf{k}(z)B_\zeta(zI - A_\zeta)^{-1}\right] \tag{9}$$

where res denotes the *residue.* By combining (1), (8) and (9) we see that the null-pole subspace associated with F at $z = 0$ has the explicit description

$$\mathcal{O}_0^r[F]_0 = \{\mathbf{x}(zI - A_\pi)^{-1}C_\pi + \mathbf{k}(z) : \mathbf{x} \in \mathbb{C}^{n_\pi}, \mathbf{k} \in \mathcal{O}_0^r\,,\ \mathbf{x}S = res_{z=0}\left[\mathbf{k}(z)B_\zeta(zI - A_\zeta)^{-1}\right]\}, \tag{10}$$

where $((B_\zeta, A_\zeta), (A_\pi, C_\pi), S)$ is a null-pole triple for F at $z = 0$. (For a self-contained complete proof of this statement, see Theorem 12.3.1 of [4]).

It is possible to construct a canonical null-pole triple for a given $r \times r$-matrix function F meromorphic at $z = 0$. The details of this construction can be found in [4]. Following the usual convention, in case no matrix entry of F (respectively, of F^{-1}) has a pole at $z = 0$, a null-pole triple for F will be written simply as a zero pair (respectively, a pole pair).

A triple $\Upsilon = ((B_\zeta, A_\zeta), (A_\pi, C_\pi), S)$ consisting of a pair of matrices (A_π, C_π) (of sizes $n_\pi \times n_\pi$ and $n_\pi \times r$ and satisfying (6), a pair (B_ζ, A_ζ) (of sizes $r \times n_\zeta$ and $n_\zeta \times n_\zeta$) satisfying (4) and with S satisfying (3) will be called a rank r *admissible triple.* Given a rank r admissible triple Υ there is an $r \times r$-matrix function F meromorphic at $z = 0$ such that Υ is the null-pole triple of F at $z = 0$.

If U and V are invertible matrices of appropriate size, then the triple

$$\widetilde{\Upsilon} = ((\,B_\zeta U, U^{-1}A_\zeta U), (V^{-1}A_\pi V, V^{-1}C_\pi), V^{-1}SU) \tag{11}$$

is also a null-pole triple for F . One says the null-pole triples Υ and $\widetilde{\Upsilon}$ are similar. If Υ is an admissible triple, then the collection of $\mathcal{S}(\Upsilon)$ of triples of the form (11), where U and V vary over invertible matrices of the appropriate size, will be called the similarity orbit of Υ. An important result from [6] establishes a one-to-one correspondence between the similarity orbits of admissible rank r triples and rank r local matrix divisors at $z=0$ in $\mathbf{C}$. This correspondence is a consequence of the following result: Let F_1 and F_2 be regular $r\times r$-meromorphic matrix functions defined in a neighborhood of $z=0$. The matrix functions F_1 and F_2 are associated with similar null-pole triples if and only if for some invertible analytic $r\times r$-matrix function H, $F_2 = HF_1$ in a neighborhood of $z=0$. Thus if $\mathcal{S}(\Upsilon)$ is the similarity orbit of a rank r admissible triple, there is a unique rank r local matrix divisor Θ_0 at $z=0$ associated with $\mathcal{S}(\Upsilon)$. This divisor Θ_0 consists of the set of germs at $z=0$ of regular $r\times r$-meromorphic matrix functions F such that every element in $\mathcal{S}(\Upsilon)$ is a null-pole triple of F.

At a point p on a Riemann surface, it is possible to specify a local matrix divisor Θ_p using the similarity orbit $\mathcal{S}(\Upsilon)$ of an admissible triple together with local coordinates (s,V), where s maps the neighborhood V of p into $\mathbf{C}$ with $s(p)=0$. This matrix divisor consists of the collection of germs of regular $r\times r$-meromorphic matrix functions L that have the form $L(q)=F(s(q))$ in a neighborhood of p, where F admits $\mathcal{S}(\Upsilon)$ as a set of null-pole triples at $s=0$. A less concrete but coordinate free approach to the null-pole triple can be given as follows. The value of the matrix divisor Θ_p at a point $p\in M$ defines the null-pole subspace $\mathcal{S}(\Theta_p)$ of the space $\mathcal{M}_p^r$ of $r-$dimensional meromorphic row vector germs. Introduce the pole space $\mathcal{P}_p = [\mathcal{S}(\Theta_p)+\mathcal{O}_p^r]/\mathcal{O}_p^r$ and the null space $\mathcal{N}_p = \mathcal{O}_p^r/[\mathcal{O}_p^r\bigcap\mathcal{S}(\Theta_p)]$. The spaces $\mathcal{P}_p$ and $\mathcal{N}_p$ are finite dimensional. The triple $(\mathcal{N}_p, \mathcal{P}_p, T)$, where $T:\mathcal{N}_p\to\mathcal{P}_p$ is a linear transformation can be used as an alternative to the null-pole triple introduced above. See, e.g., [15],[5].

2. Interpolation Problems

The concrete prescription of null-pole triple data in interpolation problems will be given in fixed local coordinates at the interpolation nodes. Let $z_1,\ldots,z_K$ be fixed points on the Riemann surface M and (s_j,V_j) be local coordinates at z_j with $s_j(z_j)=0$, $j=1,\ldots,K$. Let

$$\Upsilon_j = ((B_{\zeta_j},A_{\zeta_j}),(A_{\pi_j},C_{\pi_j}),S_j)\ j=1,\ldots,K$$

be rank r admissible triples. We do allow the possibility that Υ_j consists only of a zero or pole pair. The collection

$$\mathcal{D}:\{(s_1,z_1,\Upsilon_1),\ldots,(s_K,z_K,\Upsilon_K)\} \tag{12}$$

will be referred to as an admissible rank r interpolation data set.

First Basic Interpolation Problem: Given the admissible rank r interpolation data set (12), determine whether there exists a regular meromorphic function G on M such that in a neighborhood of z_j

$$G(p) = F_j(s_j(p)),$$

where F_j is a regular $r \times r$-meromorphic matrix function at $s = 0$ having Υ_j as a null-pole triple for $j = 1, \ldots, K$ and such that at other points of M, the matrix function G is a non-singular analytic matrix function.

A solution of the *First Basic Interpolation Problem* will be presented below.

In order to discuss multiple valued solutions of our interpolation problems, it is natural to work in the environment of the universal cover. To this end let $\rho : \widetilde{M} \to M$ be "the" universal cover of M. For convenience whenever local coordinates (s, V) are chosen at a point $p \in M$ it will be assumed that $\rho^{-1}(V)$ is a disjoint collection of neighborhoods of points in $\rho^{-1}(\{p\})$ and, therefore, $s \circ \rho$ provides local coordinates at points in $\rho^{-1}(\{p\})$. The data

$$\rho^*\mathcal{D} : \ \{(s_1 \circ \rho, \rho^{-1}(\{z_1\}), \Upsilon_1), \ldots, (s_K \circ \rho, \rho^{-1}(\{z_K\}), \Upsilon_K)\} \tag{13}$$

will be called admissible rank r interpolation data on $\widetilde{M}$.

Second Basic Interpolation Problem: Given the admissible rank r interpolation data set (13) on $\widetilde{M}$ determine whether there exists a regular $r \times r$-meromorphic matrix function G with constant matrix factor of automorphy on $\widetilde{M}$ such that in a neighborhood of a point in $\rho^{-1}(\{p_j\})$

$$G(p) = F_j(s_j(\rho(p)))$$

where F_j is a regular $r \times r$-meromorphic matrix function at $s = 0$ having Υ_j as a null-pole triple for $j = 1, \ldots, K$ and such that at other points of $\widetilde{M}$, the matrix function G is a non-singular analytic matrix function.

A solution G to the *Second Basic Interpolation Problem* will be called a *flat solution.*

Note that these interpolation problems only depend on the similarity orbits $\mathcal{S}(\Upsilon_j)$ of the admissible triples Υ_j, $j = 1, \ldots, K$ and the data $\mathcal{D}$ given in (12) can be taken in the form

$$\mathcal{SD} : \{(s_1, z_1, \mathcal{S}(\Upsilon_1)), \ldots, (s_K, z_K, \mathcal{S}(\Upsilon_K))\}. \tag{14}$$

Indeed, it is important to recognize that the data $\mathcal{SD}$ given in (14) determines a unique matrix divisor $\Theta_{\mathcal{SD}}$ on M and, conversely, once coordinates are fixed at points in its support a matrix divisor Θ on M determines a unique set of data $\mathcal{SD}_\Theta$ of the form (14).

It is necessary to assemble interpolation data as follows: Let $z_1, \ldots, z_{N_0}$ be a list of the points z_j, where a zero pair appears in some Υ_j and $w_1, \ldots, w_{N_\infty}$ a list of the points z_j where a pole pair appears in some Υ_j. One will have $z_i = w_j$ for a pair (i, j) whenever there is a coupling matrix at $z_i = w_j$. In the sequel, we will frequently assume the data set has been split into three cases corresponding to "zero only", "pole only" or "pole-zero coupling." The points $z_1, \ldots, z_{N_0^0}$ will denote the interpolation points $z_1, \ldots, z_{N_0}$ where the data consists only of a zero pair and $z_{N_0^0+1}, \ldots, z_{N_0^0}$ the interpolation points $z_1, \ldots, z_{N_0}$ where there is a nontrivial coupling matrix; $w_1, \ldots, w_{N_\infty^0}$ will denote the interpolation points $w_1, \ldots, w_{N_\infty}$ where the data consists only of a pole pair and $w_{N_\infty^0+1}, ..., v_{N_\infty}$ will be a list of the interpolation points where there is a nontrivial coupling matrix. Obviously, $N_c = N_0 - N_0^0 = N_\infty - N_\infty^0$ and it can be assumed that $z_{N_0^0+j} = w_{N_\infty^0+j}$, $j = 1, \cdots, N_c$. Local coordinates at z_j will be denoted by t_j, $j = 1, ..., N_0$ and at w_i, by s_i, $i = 1, ..., N_\infty$. In addition, whenever $w_i = z_j$ we will take the local parameters to coincide: $s_i = t_j$. In the sequel, we will sometimes drop the subscripts and write $s = s(p)$ for the fixed coordinates at a node. To avoid confusion about which index is used for the associated coupling matrix, we write S_{ij} for the coupling matrix associated with points $w_i = z_j$.

3. Vector Bundles and Interpolation Data

A rank r matrix divisor or a collection of admissible rank r interpolation data corresponds in a natural way to a rank r complex vector bundle over the Riemann surface M. We briefly describe this correspondence. First, suppose that Θ is a rank r matrix divisor and $\{V_\alpha\}_{\alpha \in A}$ is an open cover of M with the property that there is a regular $r \times r$-meromorphic matrix function L_α on V_α such that $[L_a]_p$ belongs to the value of Θ at $p \in M$. The invertible holomorphic 1-cocycle $\{\Phi_{\alpha\beta}\}_{(\alpha,\beta)\in A\times A}$ given by

$$\Phi_{\alpha\beta}(p) = L_\alpha(p) L_\beta^{-1}(p),\ p \in V_\alpha \cap V_\beta \tag{15}$$

defines a rank r vector bundle E_Θ over M. Indeed, by this construction Θ corresponds to a well-defined class e_Θ of holomorphically equivalent vector bundles over M. Further, $e_\Theta = e_{\widetilde{\Theta}}$ if and only if the divisors Θ and $\widetilde{\Theta}$ are linearly equivalent. On the other hand, given admissible rank r interpolation data $\mathcal{SD}$ as in (14) one can find local solutions L_α to the interpolation problem with this data on domains V_α, where $\{V_\alpha\}_{\alpha\in A}$ covers M. The cocycle (15) defines a rank r vector bundle $E_{\mathcal{SD}}$ on M. The corresponding equivalence class of bundles will be denoted $e_{\mathcal{SD}}$. Using the above notations one has

$$e_\Theta = e_{\mathcal{SD}_\Theta} \text{ and } e_{\mathcal{SD}} = e_{\Theta_{\mathcal{SD}}}.$$

It is easy to see that the *First Basic Interpolation Problem* with data $\mathcal{D}$ has a solution if and only if the bundle $E_{\mathcal{SD}}$ is holomorphically equivalent to the trivial bundle. The *Second Basic Interpolation Problem* has a solution if and only if the bundle $E_{\mathcal{SD}}$ is holomorphically equivalent to a flat bundle.

The degree of a vector bundle V is by definition the degree of the associated determinant line bundle $\det V$ (i.e., the line bundle with transition functions $\{\det \Phi_{\alpha\beta}\}$ where $\{\Phi_{\alpha\beta}\}$ are the transition functions for V). The degree of a line bundle in turn is the number of zeros minus the number of poles of any holomorphic section. One can show that the degree of a bundle of the form $E_{\mathcal{SD}}$ is the integer

$$d = d_{\mathcal{SD}} = \sum_{j=1}^{K} (n_{\pi_j} - n_{\zeta_j}),$$

where A_{π_j} is of size $n_{\pi_j} \times n_{\pi_j}$ and A_{ζ_j} is of size $n_{\zeta_j} \times n_{\zeta_j}$, $j = 1, \ldots, K$. This follows from the connection between the null-pole triple Υ and the local Smith-McMillan form for an associated interpolant L_α (see Theorem 3.1.2 of [4]). A flat bundle necessarily has degree zero.

In the sequel it will be important to consider line bundles of degree $g-1$ which have no holomorphic sections; such bundles are characterized explicitly by the fact that their image under the Abel-Jacobi map (appropriately translated) does not lie on the divisor of the classical Riemann theta function. If we fix a base point p_0 in M, these line bundles correspond to divisors λ of the form

$$\lambda = p_1 + \cdots + p_g - p_0$$

where $\mu = p_1 + \cdots + p_g$ is a non-special divisor in the g-fold symmetric product $M^{(g)}$. The notation $\mathbb{L}_\lambda$ will be used for the line bundle corresponding to the divisor λ. For the most part we will only consider degree $g-1$ line bundles of the above form with $h^0(\mathbb{L}_\lambda) = 0$. This latter condition means that there is no nonconstant meromorphic function with poles only in μ. Any degree $g-1$ line bundle $\mathbb{L}$ satisfying $h^0(\mathbb{L}) = 0$ will be called a *non-special line bundle.* The significance of such line bundles can be seen in the following result:

Proposition 3. *Let E be a complex vector bundle of degree zero on the closed Riemann surface M of genus g. A sufficient condition that for E to be flat is the existence of a non-special line bundle $\mathbb{L}$ of degree $g-1$such that $h^0(\mathbb{L} \otimes E) = 0$. In the case where $g = 1$, this condition is also necessary.*

Proof. Assume $E = E_1 \oplus \cdots \oplus E_J$ is a decomposition of E into indecomposable bundles E_i of rank r_i, $i = 1, ..., J$. Then $h^0(\mathbb{L} \otimes E_i) = 0$, $i = 1, ..., J$. By the Riemann-Roch Theorem

$$-h^1(\mathbb{L} \otimes E_i) = r_i(g-1) + \deg E_i + r_i(1-g) = \deg E_i, \;\; i = 1, ..., J.$$

Since, $\deg E_1 + \cdots + \deg E_J = 0$ we conclude $\deg E_1 = \cdots = \deg E_J = 0$. The classical result of Weil (see, e.g., [**?**])implies that the bundle E is flat. In the case where $g = 1$, the result of Atiyah descibed earlier gives the representation of E in the form

$$E = \mathbb{L}_1 \otimes F_{h_1} \oplus ... \oplus \mathbb{L}_s \otimes F_{h_s}$$

where the line bundles $\mathbb{L}_1, \cdots, \mathbb{L}_s$ have degree zero. It is easy to see that there are line bundles $\mathbb{L}$ of degree zero such that $h^0(\mathbb{L} \otimes \mathbb{L}_i) = 0$, $i = 1, ..., s$. With such a choice of $\mathbb{L}$, $h^0(\mathbb{L}{\otimes}E) = 0$. This completes the proof.

Remark 1. *It is not hard to construct examples where $g > 1$ and where the converse of the result in this last proposition doesn't hold. Let L_ξ, L_η be line bundles on a compact Riemann surface of genus $g \geq 4$ that satisfy*

$$0 < \deg L_\xi = -\deg L_\eta : h^1(L_\xi L_\eta^{-1}) \neq 0.$$

For example, L_ξ could be the line bundle corresponding to the divisor $\xi = p$ consisting of a single point p and L_η the line bundle corresponding to the divisor $\eta = -p_1 - p_2 + z_1$, for distinct points p_1, p_2, z_1 distinct from p. It follows from the Riemann-Roch Theorem that $h^1(L_\xi L_\eta^{-1}) \neq 0$. Using a non-zero element σ from $H^1(L_\xi L_\eta^{-1})$ one constructs a indecomposable rank 2 vector bundle E using the transition matrices

$$\Phi_{\alpha\beta} = \begin{bmatrix} \xi_{\alpha\beta} & \sigma_{\alpha\beta} \\ 0 & \eta_{\alpha\beta} \end{bmatrix},$$

where $\{\xi_{\alpha\beta}\}$ and $\{\eta_{\alpha\beta}\}$ are transition functions for the bundles L_ξ and L_η relative to a suitable cover of M. Let L_λ be a line bundle of degree $g-1$. Then $\lambda \otimes E$ has a "triangular" form with 1,1-entry $L_{\lambda\xi}$. Since this line bundle has sections, we conclude that $h^o(L_\lambda \otimes E) \neq 0$, for any line bundle L_λ of degree $g-1$.

Remark 2. *In case $g > 2$, there are examples of semi-stable degree zero bundles E satisfying $h^0(\mathbb{L} \otimes E) > 0$, for every non-special line bundle $\mathbb{L}$ of degree $g-1$. Recall that a bundle E is called* semi-stable *in case*

$$\mu_F \equiv \frac{\deg F}{rank F} \leq \mu_E = \frac{\deg E}{rank E}$$

for all subbundles $F \subset E$ (see, [11],[13]). Every semi-stable bundle of degree zero is flat. This again follows from the aforementioned result of Weil. Indeed, if E is semi-stable of degree zero, then $\deg E_i \leq 0$ for each summand in the decomposition $E = E_1 \oplus \cdots \oplus E_J$, where $E_1, ..., E_J$ are indecomposable. Since $\sum_{i=0}^{J} \deg E_i = \deg E = 0$, we have $\deg E_i = 0$, $i = 1, ..., \dot{J}$, and, consequently, E is flat. An example, of a semi-stable bundle E of degree zero such that $h^0(\mathbb{L} \otimes E) > 0$ for all degree $g-1$ line bundles $\mathbb{L}$ can be found in [10].

Remark 3. *In the case M has genus one, every flat bundle E is semi-stable. This follows from the representation $E = \mathbb{L}_1 \otimes F_{h_1} \oplus ... \oplus \mathbb{L}_s \otimes F_{h_s}$ of Atiyah and the fact that direct sums of semi-stable bundles with the same slope are semi-stable (see, e.g. [13]).*

4. The Flat Case

Let w be a fixed point of M and (s, V) be fixed local coordinates at w, where as usual we assume $s(w) = 0$. Then with λ satisfying $h^0(\mathbb{L}_\lambda) = 0$,for any $k \geq 0$ an integer, there is a unique meromorphic function f^λ_{kw} whose divisor satisfies $(f^\lambda_{kw}) + \lambda + kw \geq 0$ and such that in the coordinate s, this function f^λ_{kw} has principal Laurent part at $s = 0$ equal to $\frac{1}{s^k(p)}$. To see this, note that the dimension of the space of meromorphic functions f whose divisor satisfies $(f) + \lambda + kw \geq 0$, which is equal to the dimension $h^0(\mathbb{L}_{\lambda+kw})$ of the space of holomorphic sections of the line bundle $\mathbb{L}_{\lambda+kw}$ corresponding to the divisor $\lambda + kw$, equals k. To see this note first from the Riemann-Roch theorem $h^0(\mathbb{L}_{\lambda+kw}) \geq \deg(\lambda + kw) - (g - 1) = k$. On the other hand, it can be easily verified from the assumption $h^0(\mathbb{L}_\lambda) = 0$,that $h^0(\mathbb{L}_{\lambda+kw}) \leq k$. Thus there exists a one-dimensional space of meromorphic functions f with $(f) + \lambda + kw \geq 0$ with exactly a k^{th}-order pole at w; by normalizing the principal part, we obtain a uniquely defined function f^λ_{kw}.

Suppose that A is the $n \times n$ Jordan cell

$$\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \cdots & \cdots & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix}.$$

We introduce the $n \times n$ matrix function

$$f^\lambda_{w,A} = \begin{bmatrix} f^\lambda_w & f^\lambda_{2w} & \cdots & f^\lambda_{nw} \\ 0 & f^\lambda_w & \ddots & \vdots \\ \vdots & \ddots & \ddots & f^\lambda_{2w} \\ 0 & \cdots & 0 & f^\lambda_w \end{bmatrix}.$$

We extend this definition to an arbitrary nilpotent matrix by following the conventions

$$f^\lambda_{w,SAS^{-1}} = Sf^\lambda_{w,A}S^{-1} \textit{ and } f^\lambda_{w,A_1 \oplus A_2} = f^\lambda_{w,A_1} \oplus f^\lambda_{w,A_2},$$

where, S is an invertible matrix. Then for any nilpotent matrix N, the difference

$$f^\lambda_{w,N}(p) - (s(p)I - N)^{-1}$$

is analytic at $p = w$. Note this last identity reflects the fact that the local (left) pole pair of $f^{\lambda}_{w,N}$ at w in the coordinate s has the form $(C_{\pi}, A_{\pi}) = (I_r, N)$, where N is $r \times r$.

We will provide a concrete description of the space of holomorphic sections of the bundle

$$\mathbb{L}_{\lambda} \otimes E^*_{\mathcal{D}}$$

where $\mathbb{L}_{\lambda}$ is a non-special degree $g-1$ line bundle and $E^*_{\mathcal{D}}$ is the adjoint of the bundle $E_{\mathcal{D}}$. The divisor λ determining the line bundle $\mathbb{L}_{\lambda}$ will be assumed to have the form $\lambda = p_1 + \cdots + p_g - p_0$, where the points $p_0, p_1, \ldots, p_g$ are distinct from the interpolation nodes and $\mu = p_1 + \ldots + p_g$ is a non-special divisor.

Let $\{\Phi_{\alpha\beta}\}$ is a holomorphic cocycle determining a vector bundle E and $\Phi_{\alpha\beta} = L_{\alpha} L_{\beta}^{-1}$ a trivialization of E by a family $\{L_{\alpha}\}_{\alpha \in A}$ of regular $r \times r$-meromorphic matrix functions relative to the open cover $\{V_{\alpha}\}_{\alpha \in \mathcal{A}}$. We introduce the collection of vector-valued meromorphic functions of the form

$$L^{\vee}(E) = \{\mathbf{h} \in \mathcal{M}^r(M) : [\mathbf{h}]_p \in \mathcal{O}^r_p[L_{\alpha}]_p,\ p \in V_{\alpha}\}.$$

Note that the condition $[\mathbf{h}]_p = [\mathbf{h}_{\alpha}]_p [L_{\alpha}]_p$, where $\mathbf{h}_{\alpha}$ is holomorphic "transposes" to the condition $(L^t_{\alpha})^{-1} L^t_{\beta} \mathbf{h}^t_{\beta} = \mathbf{h}^t_{\alpha}$, ("$t$" denotes the transpose operation) on the intersection $V_{\alpha} \bigcap V_{\beta}$. Thus the space $L^{\vee}(E)$ is naturally isomorphic to the space $\Gamma(E^*)$ of holomorphic sections of the adjoint bundle E^* determined by the cocycle $\{(\Phi^t_{\alpha\beta})^{-1}\}$.

In order to separate the role of λ, we will assume that $\mathbb{L}_{\lambda}$ and $E_{\mathcal{D}}$ are trivialized separately relative to the open cover $\{V_{\alpha}\}_{\alpha \in \mathcal{A}}$. That is, it will be assumed that $k_{\alpha}, \alpha \in \mathcal{A}$ is a family of scalar meromorphic functions which interpolates the divisor $\lambda = p_1 + \cdots + p_g - p_0$ and that $L_{\alpha}, \alpha \in \mathcal{A}$ is a family of $r \times r$ meromorphic matrix functions which locally interpolates the divisor $\mathcal{D} : \{(s_1, z_1, \Upsilon_1), \ldots, (s_K, z_K, \Upsilon_K)\}$. The notation L^{λ}_{α} will be used for the matrix functions $L^{\lambda}_{\alpha} = k^{-1}_{\alpha} L_{\alpha}, \alpha \in \mathcal{A}$.

We first claim that the elements in $L^{\vee}(E^{\lambda}_{\mathcal{D}})$ necessarily have the form

$$\mathbf{h} = \sum_{i=1}^{N_{\infty}} \mathbf{u}_j f^{\lambda}_{w_i, A_{\pi_i}} C_{\pi_i}, \tag{16}$$

where $\mathbf{u}_i$ is an n_{π_i}-dimensional row vector for $i = 1, \ldots., N_{\infty}$. One sees this last claim as follows: Let $\mathbf{h}$ be an element of $L^{\vee}(E^{\lambda}_{\mathcal{D}})$. Then for each point w_i, we know that $[\mathbf{h}]_{w_i}$ is in the null-pole subspace $\mathcal{O}^r_{w_i}[L^{\lambda}_{\alpha}]_{w_i}$. From the formula (10) we see that there is a (necessarily unique) vector $\mathbf{u}_i$ such that $\mathbf{h}(s^{-1}(z))$ and $\mathbf{u}_i(zI - A_{\pi_i})^{-1} C_{\pi_i}$ have the same principal part at $z = 0$ (here s denotes local coordinates at $p = w_i$). This implies $\mathbf{h} - \mathbf{u}_i f^{\lambda}_{w_i, A_{\pi_i}} C_{\pi_i}$ has an analytic continuation to $p = w_i$. If we set

$\mathbf{g} = \sum_{i=1}^{N_\infty} \mathbf{u}_i f^\lambda_{w_i, A_{\pi_i}} C_{\pi_i}$, then it follows that $\mathbf{h} - \mathbf{g}$ has analytic continuation through each of the points $w_1, \ldots, w_{N_\infty}$. From the construction, it is now easy to check that each scalar component $h_k - g_k$ $(k = 1, ..., r)$ of the vector function $\mathbf{h} - \mathbf{g}$ has divisor $(h_k - g_k)$ satisfying

$$(h_k - g_k) + \lambda \geq 0. \tag{17}$$

Since $h^0(\mathbb{L}_\lambda) = 0$, we conclude that $\mathbf{h} = \mathbf{g}$ and, therefore, $\mathbf{h}$ has the claimed form.

The following lemmas will be used to complete the description of $L^\vee(E^\lambda_{\mathcal{D}})$.

Lemma 4. *Suppose* $w_i \neq z_j$, $j = 1, ..., N_o$. *If* $\mathbf{h} \in \mathcal{M}^r$, *then* $[\mathbf{h}]_{w_i} \in \mathcal{O}^r_{w_i}[L^\lambda_\alpha]_{w_i}$ *if and only if*

$$[\mathbf{h}]_{w_i} - [\mathbf{u}_i f^\lambda_{w_i, A_{\pi_i}} C_{\pi_i}]_{w_i}, \tag{18}$$

has analytic continuation at $p = w_i$ *for some row vector* $\mathbf{u}_i \in \mathbf{C}^{n_{\pi_i}}$.

Proof. This result follows immediately from (10) and the fact that $f^\lambda_{w_i, A_{\pi_i}}(p) - (s_i(p)I - A_{\pi_i})^{-1}$ is analytic at w_i.

Before stating the next lemma we introduce the notations

$$\Gamma^\lambda_{ij} = -res_{z=0}[f^\lambda_{w_i, A_{\pi_i}}(t_j^{-1}(z)) C_{\pi_i} B_{\zeta j}(z - A_{\zeta_j})^{-1}], \ \textit{for all } i, j \textit{ with } z_j \neq w_i$$

$$\Gamma^\lambda_{ij} = S_{ij} - res_{z=0}[\{f^\lambda_{w_i, A_{\pi_i}}(s_i^{-1}(z)) - (z - A_{\pi_i})^{-1}\} C_{\pi_i} B_{\zeta_j}(z - A_{\zeta j})^{-1}], \ \textit{for } i, j \textit{ with } z_j = w_i$$

$$\Gamma^\lambda = \left[\Gamma^\lambda_{ij}\right]_{N_0 \times N_\infty}. \tag{19}$$

Lemma 5. *If* $\mathbf{h}$ *has the form (16), then for* $1 \leq j \leq N_0$, *the row vector function* $[\mathbf{h}]_{z_j}$ *is in* $\mathcal{O}^r_{z_j}[L^\lambda_\alpha]_{z_j}$ *if and only if*

$$\sum_{i=0}^{N_\infty} \mathbf{u}_i \Gamma^\lambda_{ij} = \mathbf{0}.$$

Proof. The arguments for the cases $1 \leq j \leq N_0^0$ and $N_0^0 + 1 \leq j \leq N_0$ are similar. We will only present the details for the latter case. Suppose j satisfies $N_0^0 + 1 \leq j \leq N_0$. Without loss of generality assume that $w_1 = z_j$. Write $\mathbf{h}$ in the form

$$\begin{aligned}
\mathbf{h}(s_1^{-1}(z)) &= \mathbf{u}_1 f^\lambda_{w_1, A_{\pi_1}}(s_1^{-1}(z)) C_{\pi_1} + \sum_{i=2}^{N_\infty} \mathbf{u}_i f^\lambda_{w_i, A_{\pi_i}}(s_1^{-1}(z)) C_{\pi_i} \\
&= \mathbf{u}_1 (z - A_{\pi_1})^{-1} C_{\pi_1} + \mathbf{u}_1 [f^\lambda_{w_1, A_{\pi_1}}(s_1^{-1}(z)) - (z - A_{\pi_1})^{-1}] C_{\pi_1} \\
&\quad + \sum_{i=2}^{N_\infty} \mathbf{u}_i f^\lambda_{w_i, A_{\pi_i}}(s_1^{-1}(z)) C_{\pi_i} \\
&= \mathbf{u}_1 (z - A_{\pi_1})^{-1} C_{\pi_1} + \mathbf{k}(z),
\end{aligned}$$

where $\mathbf{k}$ is analytic at $z = 0$. Using the description of $\mathcal{O}^r_{z_i}[L^\lambda_\alpha]_{z_i}$ in (10) we learn that $\mathbf{h}$ lies in this subspace if and only if

$$\begin{aligned}\mathbf{u}_1 S_{1j} &= res_{z=0}(\mathbf{k}(z)B_{\zeta_j}(z - A_{\zeta_j})^{-1}) \\ &= \mathbf{u}_1 res_{z=0}[\{f^\lambda_{w_1,A_{\pi_1}}(s_1^{-1}(z)) - (z - A_{\pi_1})^{-1}\}C_{\pi_1}B_{\zeta_j}(z - A_{\zeta_j})^{-1}] \\ &\quad + \sum_{i=2}^{N_\infty} \mathbf{u}_i res_{z=0}[f^\lambda_{w_i,A_{\pi_i}}(s_1^{-1}(z))C_{\pi_i}B_{\zeta_j}(z - A_{\zeta_j})^{-1}] \\ &= \mathbf{u}_1[S_{1j} - \Gamma^\lambda_{i1}] - \sum_{i=2}^{N_\infty} \mathbf{u}_i \Gamma^\lambda_{ij}.\end{aligned}$$

The result follows.

Theorem 6. *Let $\mathbb{L}_\lambda$ be a line bundle of degree $g - 1$ determined by the divisor $p_1 + ... + p_g - p_0$, where $\mu = p_1 + ... + p_g$ is a non-special divisor and let the pole-zero interpolation data be given by $\mathcal{D} : \{(s_1, z_1, \Upsilon_1), \ldots, (s_K, z_K, \Upsilon_K)\}$.The vector space of sections of the vector bundle $\mathbb{L}_\lambda \otimes E^*_\mathcal{D}$ is isomorphic to the collection of row vector meromorphic functions of the form $\mathbf{h} = \sum_{i=1}^{N_\infty} \mathbf{u}_i f^\lambda_{w_i,A_{\pi_i}} C_{\pi_i}$, where the row vector $\mathbf{u} = [\mathbf{u}_1, ..., \mathbf{u}_{N_\infty}]$ satisfies $\mathbf{u}\Gamma^\lambda = \mathbf{0}$. In particular, $h^0(\mathbb{L}_\lambda \otimes E^*_\mathcal{D})$ equals the dimension of the left-kernel of Γ^λ.*

Proof. The above lemmas imply that the space $L^\vee(E^\lambda_\mathcal{D})$ is isomorphic to the collection of row vector meromorphic functions of the form $\mathbf{h} = \sum_{i=1}^{N_\infty} \mathbf{u}_i f^\lambda_{w_i,A_{\pi_i}} C_{\pi_i}$, where the row vector $\mathbf{u} = [\mathbf{u}_1, ..., \mathbf{u}_{N_\infty}]$ satisfies $\mathbf{u}\Gamma^\lambda = \mathbf{0}$. The result now follows from the isomorphism between $L^\vee(E^\lambda_\mathcal{D})$ and $\Gamma(\mathbb{L}_\lambda \otimes E^*_\mathcal{D})$ which was described above. This ends the proof.

The following result represents a partial solution to the *Second Basic Interpolation Problem* and is an immediate corollary of the last theorem and Proposition 2:

Corollary 7. *Let $\mathcal{D} : \{(s_1, z_1, \Upsilon_1), \ldots, (s_K, z_K, \Upsilon_K)\}$ be given pole-zero interpolation data of degree zero on a closed Riemann surface M with genus $g \geq 1$. If there exists a divisor $\lambda = p_1 + \cdots + p_g - p_0$ of degree $g - 1$ with $h^0(\mathbb{L}_\lambda) = 0$ such that the matrix Γ^λ is invertible, then there is an $r \times r$ meromorphic function F on the universal cover $\rho : \widetilde{M} \to M$ with flat factor of automorphy interpolating the pole-zero data $\rho^*\mathcal{D}$. In the case, where $g = 1$, such a matrix function F exists if and only if Γ^λ is invertible for some non-zero degree $g - 1$divisor λ with $h^0(\mathbb{L}_\lambda) = 0$.*

5. THE SINGLE VALUED CASE

In this section we solve the *First Basic Interpolation Problem* which was introduced above. We begin with the remark that a necessary condition for the existence of a solution is that the bundle $E_{\mathcal{D}}$ be trivial and, consequently, for every degree $g-1$ divisor λ with $h^0(\mathbb{L}_\lambda)=0$, we have $h^0(\mathbb{L}_\lambda \otimes E^*_{\mathcal{D}})=0$. Thus for all such divisors λ, the matrix Γ^λ is invertible.

In order to simplify the discussion, we will assume the Riemann surface is presented as a fundamental domain R_0 on the universal cover $\rho : \widetilde{M} \to M$ and that the interpolation points $z_1, ..., z_K$ as well as points in the divisors $\lambda = p_1 + ... + p_g - p_0$ are in R_0. The functions f^λ_{kw} will be assumed to be functions in global coordinates on the universal cover $\widetilde{M}$ which can be assumed to be $\mathbb{C}$ or the unit disc $\mathbb{D}$. The pole-zero interpolation data will be taken in the form

$$\mathcal{D} : \{(z_1, \Upsilon_1), \dots, (z_K, \Upsilon_K)\},$$

where we suppress writing the coordinates $s_j(u) = u - z_j$ at the points z_j, $j = 1, ..., K$.

Fix a divisor $\lambda = p_1 + ... + p_g - p_0$ satisfying $h^0(\mathbb{L}_\lambda) = 0$, with $p_0, ..., p_g$ distinct from $z_1, ..., z_K$. If there exists a solution F of the first basic interpolation problem, it can be assumed to satisfy $F(p_0) = I_r$ and, therefore, has the form

$$F = I_r + \sum_{i=1}^{K} U_i f^\lambda_{w_i, A_{\pi_i}} C_{\pi_i}, \tag{20}$$

where U_i are $r \times n_{\pi_i}-$matrices. This last remark uses the fact that $h^0(\mathbb{L}_\lambda) = 0$. The rows of F are obviously in $\mathcal{O}^r_{z_i}[F]_{z_i}$ for $i = 1, ..., K$ and as in the proof of Theorem 4, one sees that

$$[U_1, ..., U_K]\Gamma^\lambda = [B_{\zeta_1}, ..., B_{\zeta_K}] \tag{21}$$

or, equivalently, the matrices $U_1, ..., U_K$, are given by

$$[U_1, ..., U_K] = [B_{\zeta_1}, ..., B_{\zeta_K}](\Gamma^\lambda)^{-1}.$$

The following proposition is an immediate consequence of the above discussion.

Proposition 8. *Let $\mathcal{D} : \{(z_1, \Upsilon_1), \dots, (z_K, \Upsilon_K)\}$ be given pole-zero interpolation data and $\lambda = p_1 + ... + p_g - p_0$ a divisor satisfying $h^0(\mathbb{L}_\lambda) = 0$, with $p_0, ..., p_g$ distinct from $z_1, ..., z_K$. The matrix Γ^λ is invertible if and only if there exists an $r \times r-$meromorphic matrix function satisfying:*

- *F and F^{-1} are analytic off $\{z_1, ..., z_K, p_0, ..., p_g\}$ with $F(p_0) = I_r$.*

- *F interpolates the divisor $\mathcal{D}$ at the points $z_1, ..., z_K$.*
- *The entries of F have at most simple poles at $p_1, ..., p_g$.*

Further, when the matrix Γ^λ is invertible, the unique F satisfying these last conditions is given by (20) with the matrices $U_1, ..., U_K$ given by (21).

In order that the matrix function F described in the preceding proposition interpolate only the data $\mathcal{D}$ one must ensure that the residues of F at $p_1, ..., p_g$ be zero. This involves additional linear conditions on the matrices $U_1, ..., U_K$ which we now describe. Introduce the notation

$$R_{ij} = res_{u=p_j}[f^\lambda_{w_i, A_{\pi_i}}(u) C_{\pi_i}], \ i = 1, ..., K; j = 1, ..., g.$$

The matrix function F given by (20) has the property that $res_{u=p_j}[F(u)]$,$j = 1, ..., g$ is the zero matrix if and only if

$$\begin{bmatrix} U_1 & \cdots & U_K \end{bmatrix} \begin{bmatrix} R_{11} & \cdots & R_{1g} \\ \vdots & & \vdots \\ R_{K1} & \cdots & R_{Kg} \end{bmatrix} = UR = B(\Gamma^\lambda)^{-1} R = 0,$$

where we are using the notations

$$R = \begin{bmatrix} R_{11} & \cdots & R_{1g} \\ \vdots & & \vdots \\ R_{K1} & \cdots & R_{Kg} \end{bmatrix}, \ U = \begin{bmatrix} U_1 & \cdots & U_K \end{bmatrix} \text{ and } B = \begin{bmatrix} B_{\zeta_1} & \cdots & B_{\zeta_K} \end{bmatrix}. \quad (22)$$

The following theorem is the main result of this section. It represents a generalization of the genus one result in [4] and also generalizes results in [2] and [8] to the case where the poles and zeros have multiplicity.

Theorem 9. *Let $\mathcal{D} : \{(z_1, \Upsilon_1), \ldots, (z_K, \Upsilon_K)\}$ be a given divisor on M. There is an $r \times r-$meromorphic matrix function F interpolating $\mathcal{D}$ if and only if for some degree $g-1$ divisor λ with $h^0(\mathbb{L}_\lambda) = 0$ the matrix Γ^λ given in (4) is invertible and $B(\Gamma^\lambda)^{-1}R = 0$, where the matrices R and B are given by (22). In this case the unique solution F of the interpolation problem satisfying $F(p_0) = I_r$ is given by (20).*

6. THE CASE OF GENUS 1

As mentioned in the introduction, in the case of simple data, the matrix Γ^λ has a nice form when M is of genus 1 and is realized in the form $M = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, where

$\operatorname{Im}\tau > 0$. In this section, we will give more details on this representation of Γ^λ when M has genus 1.

The divisor data $\mathcal{D}$ will be collected as follows. Let $z_1, \cdots, z_N; \zeta_1, \cdots, \zeta_K; w_1, \cdots, w_N$ be distinct points in the complex plane lying in a fundamental domain R_0 ($= \{u = x + i\tau y : 0 \le x, y \le 1\}$) for $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. We will write the data $\mathcal{D}$ in the form

$$\mathcal{D} : (z_1, \underline{b}_1), \cdots, (z_N, \underline{b}_N); (w_1, \mathbf{c}_1), \cdots, (w_N, \mathbf{c}_N); (\zeta_1, (\underline{\beta}_1, \boldsymbol{\gamma}_1, S_1)), \cdots, (\zeta_1, (\underline{\boldsymbol{\beta}}_K, \boldsymbol{\gamma}_K, S_K))$$

where the data $(z_1, \underline{b}_1), \cdots, (z_N, \underline{b}_N)$ consists of only simple right zero data, $(w_1, \mathbf{c}_1), \cdots, (w_N, \mathbf{c}_N)$ consists of only simple left pole data and $(\zeta_1, (\underline{b}_{N+1}, \mathbf{c}_{N+1}, S_1)), \cdots, (\zeta_1, (\underline{b}_{N+K}, \mathbf{c}_{N+K}, S_K))$ consists of data at points $\zeta_1, \cdots, \zeta_K$, where we have pole zero coupling numbers $S_1, ..., S_K$. In this notation, we supress writing the 1×1 zero matrices A_{ζ_i} and A_{π_j}. Moreover, $\mathbf{c}_j \underline{b}_j = 0$, $j = N+1, ..., N+K$.

The $r \times r$-meromorphic function $F = F(u)$ on $\mathbb{C}$ will solve the interpolation problem with data $\mathcal{D}$ in case:

- The matrix function F is holomorphic and invertible off $\{z_1, \cdots, z_N, w_1, \cdots, w_N, \zeta_1, \cdots, \zeta_K\} + \mathbb{Z} + \tau\mathbb{Z}$.
- The only poles of entries of the matrix function F are at most simple poles at points in $\{w_1, \cdots, w_N, \zeta_1, \cdots, \zeta_K\} + \mathbb{Z} + \tau\mathbb{Z}$.
- The matrix function F is holomorphic at $z_i + \mathbb{Z} + \tau\mathbb{Z}$ and $\underline{b}_i$ spans $\ker[F]$ at these points.
- The matrix function F^{-1} is holomorphic at $w_j + \mathbb{Z} + \tau\mathbb{Z}$ and $\mathbf{c}_j$ spans the left kernel of F^{-1} at these points.
- At $u = \zeta_i + \mathbb{Z} + \tau\mathbb{Z}$, the singular subspace $\mathcal{O}_u^r[F]_u$ has the description

$$\mathcal{O}_u^r[F]_u = \{\frac{\mu \mathbf{c}_{N+i}}{z-u} + \mathbf{k}(z) : \mu \in \mathbb{C}, \mathbf{k} \in \mathcal{O}_u^r \text{ such that } \mu S_i = res_{z=u}\left[\frac{\mathbf{k}(z)\underline{b}_{N+i}}{z-u}\right]\}.$$

Given distinct p_0, p_1 in the fundamental domain R_0 the divisor $\lambda = p_1 - p_0$ satisfies $h^0(\mathbb{L}_\lambda) = 0$. Indeed, by fixing p_0 and varying p_1, the divisors $\lambda = p_1 - p_0$ realize every degree zero divisor λ with $h^0(\mathbb{L}_\lambda) = 0$. The functions f_w^λ have a very explicit form in terms of the function

$$\theta_*(u) = \sum_{n \in \mathbb{Z}} \exp\left\{2\pi i[\frac{1}{2}(n+\frac{1}{2})\tau(n+\frac{1}{2}) + (n+\frac{1}{2})(u+\frac{1}{2})]\right\}.$$

Note that the function θ_* has the automorphic behavior

$$\theta_*(u+m+n\tau)=\exp\{\pi i(m-n-n^2\tau)\}\exp\{-2\pi inu\}\theta_*(u),\ m,n\in\mathbb{Z}.$$

If w in R_0 is distinct from the points p_0,p_1, for any constant C_w^λ the function

$$f_w^\lambda(u)=C_w^\lambda\frac{\theta_*(u-p_0)\theta_*(u+p_0-p_1-w)}{\theta_*(u-w)\theta_*(u-p_1)}$$

is single valued with divisor $(f_w^\lambda)=p_0+q-w-p_1$, where $q=p_1+w-p_0$. The choice

$$C_w^\lambda=\frac{\theta_*^{'}(w)\theta_*(w-p_1)}{\theta_*(w-p_0)\theta_*(p_0-p_1)}$$

normalizes f_w^λ so that it has the requisite principal part at $u=w$. The matrix Γ^λ has the form

$$\Gamma^\lambda=\begin{bmatrix}\Gamma_{11}^\lambda & \Gamma_{12}^\lambda\\ \Gamma_{21}^\lambda & \Gamma_{22}^\lambda\end{bmatrix}$$

where

$$\Gamma_{11}^\lambda=\begin{bmatrix}-f_{w_1}^\lambda(z_1)\mathbf{c}_1\underline{b}_1 & \cdots & -f_{w_1}^\lambda(z_N)\mathbf{c}_1\underline{b}_N\\ \vdots & & \vdots\\ -f_{w_N}^\lambda(z_1)\mathbf{c}_N\underline{b}_1 & \cdots & -f_{w_N}^\lambda(z_N)\mathbf{c}_N\underline{b}_N\end{bmatrix}$$

$$\Gamma_{12}^\lambda=\begin{bmatrix}-f_{w_1}^\lambda(\zeta_1)\mathbf{c}_1\underline{b}_{N+1} & \cdots & -f_{w_1}^\lambda(\zeta_K)\mathbf{c}_N\underline{b}_{N+K}\\ \vdots & & \vdots\\ -f_{w_N}^\lambda(\zeta_1)\mathbf{c}_N\underline{b}_{N+1} & \cdots & -f_{w_N}^\lambda(\zeta_K)\mathbf{c}_N\underline{b}_{N+K}\end{bmatrix}$$

$$\Gamma_{21}^\lambda=\begin{bmatrix}-f_{\zeta_1}^\lambda(z_1)\mathbf{c}_{N+1}\underline{b}_1 & \cdots & -f_{\zeta_1}^\lambda(z_N)\mathbf{c}_{N+1}\underline{b}_N\\ \vdots & & \vdots\\ -f_{\zeta_K}^\lambda(z_1)\mathbf{c}_{N+K}\underline{b}_1 & \cdots & -f_{\zeta_K}^\lambda(z_N)\mathbf{c}_{N+K}\underline{b}_N\end{bmatrix}$$

$$\Gamma_{22}^\lambda=\begin{bmatrix}S_1 & S_2-r_{\zeta_1}^\lambda(\zeta_2)\mathbf{c}_{N+1}\underline{b}_{N+2} & \cdots & S_K-r_{\zeta_1}^\lambda(\zeta_K)\mathbf{c}_{N+1}\underline{b}_{N+K}\\ S_1-r_{\zeta_2}^\lambda(\zeta_1)\mathbf{c}_{N+2}\underline{b}_{N+1} & S_2 & & S_K-r_{\zeta_2}^\lambda(\zeta_K)\mathbf{c}_{N+2}\underline{b}_{N+K}\\ \vdots & \vdots & \ddots & \vdots\\ S_1-r_{\zeta_K}^\lambda(\zeta_1)\mathbf{c}_{N+K}\underline{b}_{N+1} & S_2-r_{\zeta_K}^\lambda(\zeta_2)\mathbf{c}_{N+K}\underline{b}_{N+2} & & S_K\end{bmatrix}$$

with $r_{\zeta_i}^\lambda(u)=f_{\zeta_i}^\lambda(u)-\frac{1}{u-\zeta_i}$, $i=1,...,K$.

It follows immediately from Corollary 7 that the interpolation problem corresponding to the data $\mathcal{D}$ has a flat solution if and only if Γ^λ is invertible for some non-zero λ. In this case the zeros of $\det(\Gamma^\lambda)$ correspond precisely to the divisors $\lambda_1, ..., \lambda_s$ providing the decomposition

$$E_{\mathcal{D}} = \mathbb{L}_{\lambda_1} \otimes F_{h_1} \oplus ... \oplus \mathbb{L}_{\lambda_s} \otimes F_{h_s}$$

of Atiyah which was used earlier. Moreover, the dimension of the kernel of Γ^λ counts the number of summands in this Atiyah decomposition where $\mathbb{L}_\lambda$ appears as a factor. In particular, the bundle $E_{\mathcal{D}}$ is equivalent to a direct sum of line bundles if and only if $\det(\Gamma^\lambda)$ has r zeros (counting multiplicity) in $R_0 \backslash \{0\}$. At the other extreme, $E_{\mathcal{D}}$ is equivalent to the Atiyah bundle F_r if and only if $\det(\Gamma^\lambda)$ doesn't vanish on $R_0 \backslash \{0\}$.

In the case where there is no pole-zero coupling ($K = 0$), the $N \times N-$matrix Γ^λ has the simpler form Γ^λ_{11}, with $ij-$entry

$$\frac{\theta_*'(w_i)\theta_*(w_i - p_1)}{\theta_*(w_i - p_0)\theta_*(\lambda)} \frac{\theta_*(w_i - z_j + \lambda)\mathbf{c}_i \underline{b}_j}{\theta_*(w_i - z_j)} \frac{\theta_*(z_j - p_0)}{\theta_*(z_j - p_1)}.$$

Clearly, the zeros of $\det \Gamma^\lambda$ coincide with the zeros of the determinant of the matrix function

$$\Gamma_0^\lambda = \left[\frac{\theta_*(w_i - z_j + \lambda)\mathbf{c}_i \underline{b}_j}{\theta_*(w_i - z_j)} \right]_{N \times N}.$$

These last remarks complete the proof of Theorem 1.

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