Existence of covering spaces, 3/13/03

Let X be path connected, locally path connected, and semi-locally simply connected. Then X has a universal covering space $q : Y \to X$. Suppose that $q(y_0) = x_0$. Since the fundamental group $\pi(Y, y_0)$ is trivial, the universal cover is Galois (or "regular"), and its automorphism group equals the fundamental group of X,

$$A_q = \pi(X, x_0).$$

Let G be a subgroup of $\pi(X, x_0)$. Then G acts on the universal cover Y. Let Y/G be the orbit space of this action. For $y \in Y$, let $< y >= Gy$, the orbit of y. Let $r : Y \to Y/G$ be the quotient map, $r(y) =< y >$. Let $p : Y/G \to X$ be the map induced by q, $p(< y >) = q(y)$. Then $pr = q$ and both r and p are covering maps.

Proposition. $\pi(Y/G, < y_0 >) = G$.

Proof:

(1) $A_r = \pi(Y/G, < y_0 >)$, since Y is the universal cover of Y/G.

(2) Since $pr = q$, A_r is a subgroup of A_q. And G is also a subgroup of A_q, since $A_q = \pi(X, x_0)$. If $g \in G$ and $y \in Y$, then $gy \in < y >$ by definition of the orbit $< y >$. Thus $r(gy) = r(y)$, so $g \in A_r$. Therefore $G \subset A_r$. Now given $h \in A_r$, $hy_0 \in < y >$, so there exists $g \in G$ such that $hy_0 = gy_0$. But an automorphism of a covering space is determined by its value at one point, so $h = g$. Therefore $G = A_r$.

Statements (1) and (2) imply that $\pi(Y/G, < y_0 >) = G$, as desired.