MATH 8200, homework 2, due Thursday, January 26

1. Prove that the spaces M_g and N_g are surfaces, $g \geq 1$. Recall that M_g is the quotient space of the unit disk D^2 with symbol

$$a_1 b_1 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1},$$

and N_g is the quotient space of the unit disk with symbol

$$a_1^2 a_2^2 \cdots a_g^2.$$

In other words, prove that these spaces are compact, metrizable, and locally homeomorphic to $\mathbb{R}^2$.

2. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by $f(u, v) = (u, uv, v^2)$.

(a) Make a computer drawing of the image of f using your favorite software (Maple, Matlab, ...). Check that the derivative of f has rank 2 for $(u, v) \neq (0, 0)$, and rank 1 for $(u, v) = (0, 0)$. Show that the image of f is the set $\{(x, y, z) \mid y^2 = zx^2,\ z \geq 0\}$.

(b) The preceding example is a parametrization of the top of a crosscap by a disk. Find a parametrization of an entire crosscap by a Möbius strip.

3. Discuss the proof of Francis and Weeks' Lemma 2: A crosshandle is homeomorphic to two crosscaps. Translate their "zip" proof into a rigorous proof using quotient spaces. If you need some topological facts you don't know how to prove, at least state those facts explicitly.

4. Use the Jordan-Schönfliess Theorem (stated in the paper by Doyle and Moran) to prove that no two of the following surfaces are homeomorphic: the sphere, the torus, and the projective plane.

Here is a clearer version of the statement of the Jordan-Schönfliess Theorem:

(1) (The Jordan Curve Theorem) If C is a subset of $\mathbb{R}^2$ such that C is homeomorphic to the circle S^1, then $\mathbb{R}^2 \setminus C$ has two connected components, and C is the common boundary of these two components.
(2) (The Schönfliess Theorem) If C is a subset of $\mathbb{R}^2$ such that C is homeomorphic to the circle S^1, then there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(C) = S^1$.

Note that (2) implies (1).

Extra credit problem:

5. Prove that every compact connected 1-manifold is homeomorphic to the unit circle S^1.