

1. Find the fourth roots of $a = -2\sqrt{3} - 6i$. (You may write the answer using trig functions.)

$$|a| = \sqrt{(2\sqrt{3})^2 + 6^2} = \sqrt{48} = 4\sqrt{3}, \quad a = 4\sqrt{3}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 4\sqrt{3}\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$$

By DeMoivre's Theorem, a fourth root of a is $b = \sqrt[4]{4\sqrt{3}}\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) = \sqrt[8]{48}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

The fourth roots of unity are $1, i, -1, -i$, so the fourth roots of a are $b, bi, -b, -bi$, or $\sqrt[8]{48}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), \sqrt[8]{48}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right), \sqrt[8]{48}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right), \sqrt[8]{48}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$.

2. Let F be a field. Prove that if $f(x) \in F[x]$ is a polynomial of degree 2 or 3, then $f(x)$ is irreducible in $F[x]$ if and only if $f(x)$ has no roots in F .

Proof. Let $f(x) \in F[x]$ be a polynomial of degree 2 or 3. First we show that if $f(x)$ has a root in F , then $f(x)$ is not irreducible. Suppose $a \in F$ and $f(a) = 0$. By the Root-Factor Theorem $f(x) = (x - a)g(x)$ for some $g(x) \in F[x]$. Now $\deg f(x) = \deg(x - a) + \deg g(x) = 1 + \deg g(x)$, and $\deg f(x) > 1$ by hypothesis, so $\deg g(x) > 0$. Hence $g(x)$ is not a constant polynomial, so $f(x)$ is the product of two non-constant polynomials.

Next we show that if $f(x)$ is not irreducible, then $f(x)$ has a root in F . Suppose that $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ are non-constant polynomials with coefficients in F , i.e. $\deg g(x) \geq 1$ and $\deg h(x) \geq 1$. Now $\deg f(x) = \deg g(x) + \deg h(x)$, so if $\deg f(x) = 2$, then $\deg g(x) = 1$ and $\deg h(x) = 1$. Suppose $g(x) = ax + b$, with $a \neq 0$. Then $g(-b/a) = 0$, so $f(-b/a) = 0$, and $-b/a$ is a root of $f(x)$. If $\deg f(x) = 3$, then $\deg g(x) = 1$ and $\deg h(x) = 2$, or $\deg g(x) = 2$ and $\deg h(x) = 1$. In either case $f(x)$ has a factor $ax + b$ of degree 1, so $-b/a \in F$ is a root of $f(x)$.

3. Decide whether each of the following polynomials $f(x) \in F[x]$ is irreducible. If it is irreducible explain why. If it is not irreducible, give a factorization into irreducible factors.

(a) $x^3 + \bar{2}x + \bar{5} \in \mathbb{Z}_7[x]$

Let $f(x) = x^3 + \bar{2}x + \bar{5} \in \mathbb{Z}_7[x]$. First we test for roots in \mathbb{Z}_7 : $f(\bar{0}) = \bar{5}$, $f(\bar{1}) = \bar{1}$, $f(\bar{2}) = \bar{3}$, $f(\bar{3}) = \bar{3}$, $f(\bar{4}) = \bar{0}$, $f(\bar{5}) = \bar{0}$, $f(\bar{6}) = \bar{2}$. Thus $f(x)$ has two roots, $\bar{4}$ and $\bar{5}$. To factor $f(x)$ we divide it by $x - \bar{4} = x + \bar{3}$. The quotient is $x^2 + \bar{4}x + \bar{4}$. Now $x^2 + \bar{4}x + \bar{4} = (x + \bar{2})^2$, so we obtain the factorization $f(x) = (x + \bar{3})(x + \bar{2})(x + \bar{2}) = (x - \bar{4})(x - \bar{5})(x - \bar{5})$.

(b) $x^4 + x^3 + x + \bar{2} \in \mathbb{Z}_3[x]$

Let $f(x) = x^4 + x^3 + x + \bar{2} \in \mathbb{Z}_3[x]$. First we test for roots in \mathbb{Z}_3 : $f(\bar{0}) = \bar{2}$, $f(\bar{1}) = \bar{2}$, and $f(\bar{2}) = \bar{1}$. Thus $f(x)$ has no roots. So the only way $f(x)$ can factor into two non-constant polynomials is as the product of two polynomials of degree 2:

$$x^4 + x^3 + x + \bar{2} = (x^2 + ax + b)(x^2 + cx + d)$$

Since $f(x)$ is monic we can assume the two factors are monic. Comparing constant terms we have $bd = \bar{2}$, so we can assume $b = \bar{1}$ and $d = \bar{2}$. Now

$$(x^2 + ax + \bar{1})(x^2 + cx + \bar{2}) = x^4 + (a + c)x^3 + acx^2 + (\bar{2}a + c)x + \bar{2}.$$

Thus $a + c = \bar{1}$, $ac = \bar{0}$, and $\bar{2}a + c = \bar{1}$. Therefore $a = \bar{0}$ and $c = \bar{1}$. So we have

$$x^4 + x^3 + x + \bar{2} = (x^2 + \bar{1})(x^2 + x + \bar{2}).$$

4. True or false? Give a proof or a counterexample:

(a) $\mathbb{Q}[\sqrt{10}] \subset \mathbb{Q}[\sqrt{2}, \sqrt{5}]$

True. Since $\sqrt{2}\sqrt{5} = \sqrt{10}$, we have $\sqrt{10} \in \mathbb{Q}[\sqrt{2}, \sqrt{5}]$, so $\mathbb{Q}[\sqrt{10}] \subset \mathbb{Q}[\sqrt{2}, \sqrt{5}]$.

(b) $\mathbb{Q}[\sqrt{2}, \sqrt{5}] \subset \mathbb{Q}[\sqrt{10}]$

False. It suffices to show $\sqrt{2} \notin \mathbb{Q}[\sqrt{10}]$.

First note that $(\sqrt{10})^{2k} = 10^k$ and $(\sqrt{10})^{2k+1} = 10^k\sqrt{10}$, so if $p(x) \in \mathbb{Q}[x]$, then $p(\sqrt{10}) = a + b\sqrt{10}$ with $a, b \in \mathbb{Q}$.

Suppose $\sqrt{2} \in \mathbb{Q}[\sqrt{10}]$. Then

$$\sqrt{2} = a + b\sqrt{10},$$

with $a, b \in \mathbb{Q}$. Now $b \neq 0$ since $\sqrt{2} \notin \mathbb{Q}$. Also $a \neq 0$, for if $a = 0$ then $\sqrt{5} = 1/b$, but $\sqrt{5} \notin \mathbb{Q}$. We have

$$\begin{aligned}\sqrt{2} - a &= b\sqrt{10} \\ 2 - 2a\sqrt{2} + a^2 &= 10b^2 \\ \sqrt{2} &= \frac{10b^2 - a^2 - 2}{-2a}\end{aligned}$$

This contradicts that $\sqrt{2} \notin \mathbb{Q}$.

5. Find the splitting field F of $x^4 + x^2 + 1 \in \mathbb{Q}[x]$. Write $F = \mathbb{Q}[\alpha]$ for some $\alpha \in \mathbb{C}$.

To find the solutions of $x^4 + x^2 + 1 = 0$, first we use the quadratic formula to solve for x^2 :

$$x^2 = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

Now if $y = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$, then by DeMoivre $\sqrt{y} = \pm(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}) = \pm(\frac{1}{2} + \frac{\sqrt{3}}{2}i)$. And if $y = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$, then by DeMoivre $\sqrt{y} = \pm(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}) = \pm(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)$. Thus the four roots of $x^4 + x^2 + 1$ are

$$\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Adjoining these four roots to \mathbb{Q} we obtain the splitting field $\mathbb{Q}[\sqrt{3}i]$. (Note $\mathbb{Q}[\sqrt{3}i] \subsetneq \mathbb{Q}[\sqrt{3}, i]$).