

1. (15 points) Use the Binomial Theorem to prove the formula

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0$$

for all  $n \in \mathbb{N}$ . *Hint:*  $(1 - 1)^n = ?$ .

2. (20 points) Prove the following divisibility test. If  $n \in \mathbb{N}$ , then 11 divides  $n$  if and only if 11 divides the alternating sum of the digits of  $n$ . (For example, let  $n = 91806$ . The alternating sum of the digits of  $n$  is  $9 - 1 + 8 - 0 + 6 = 22$ , and  $11|22$ , so  $11|91806$ .)

3. (25 points) Use the Chinese Remainder Theorem to solve the following simultaneous congruence:  $x \equiv 4 \pmod{12}$ ,  $x \equiv 22 \pmod{39}$ .

4. (25 points) Let  $p$  be a prime number.

(a) Prove that if  $a \in \mathbb{N}$ , and  $p$  does not divide  $a$ , then  $\gcd(p, a) = 1$ .

(b) Use part (a) to show that every nonzero element  $\bar{a}$  of  $\mathbb{Z}_p$  has a unique inverse.

(c) Which nonzero elements of  $\mathbb{Z}_p$  are their own inverses? (Solve the equation  $\bar{a}^2 = \bar{1}$ .)

(d) Using parts (b) and (c), compute the product of all the nonzero elements of  $\mathbb{Z}_p$ . (For example, in  $\mathbb{Z}_5$  the product of all the nonzero elements is  $\bar{1} \cdot \bar{2} \cdot \bar{3} \cdot \bar{4} = \bar{24} = \bar{4}$ .)

5. (15 points) Let  $M_2(\mathbb{Z})$  be the ring of  $2 \times 2$  matrices with integer entries.

(a) Give an example of a zero-divisor in  $M_2(\mathbb{Z})$ . (Prove that it is a zero-divisor.)

(b) Give an example of a unit in  $M_2(\mathbb{Z})$ . (Prove that it is a unit.)

6. (15 points) *Extra credit problem, required for MATH 6000 students.* Prove or give a counterexample:

(a) If  $\bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}_m$  and  $\bar{a} \neq \bar{0}$ , then  $\bar{a} \cdot \bar{b} = \bar{a} \cdot \bar{c}$  implies  $\bar{b} = \bar{c}$ .

(b) If  $R$  is a ring and  $a, b, c \in R$  satisfy  $ab = 1$  and  $ca = 1$ , then  $a$  must be a unit.

(c) Every nonzero element of a commutative ring is either a unit or a zero-divisor.