MATH 4000/6000 Exam 3
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Each problem is worth 20 points.

Briefly explain your reasoning in every problem.

1. Which of the following polynomials are irreducible in $\mathbb{Q}[x]$?

   (a) $x^3 - 2x^2 + x - 5$

   (b) $x^4 + 3x + 5$

   (c) $x^5 + 6x + 15$

2. How many ideals does the ring $\mathbb{Z}_{12}$ have? List all the ideals of this ring.

3. What are the zero-divisors of the ring $\mathbb{Z}_3[x]/\langle x^2 + x\rangle$?

4. (a) If $f(x)$ is an irreducible polynomial with coefficients in a field F, explain how to construct a field K containing F so that $f(x)$ has a root in K. You may use theorems from the course to justify your answer. (Do not reprove the theorems you use.)

   (b) Show how this construction works for the polynomial $x^2 - 5 \in \mathbb{Z}_7[x]$.

5. Let $\varphi : \mathbb{Z}[i] \to \mathbb{Z}_5$ be defined by $\varphi(a + bi) = \overline{a} + 2\overline{b}$. Prove that φ is a homomorphism, and use it to prove that $\mathbb{Z}[i]/\langle 1 + 2i\rangle$ is isomorphic to $\mathbb{Z}_5$.