MATH 4000/6000 Final Exam
December 16, 2008, 8:00–11:00 am
C. McCrory

Each problem is worth 25 points, for a total of 200 points.

Briefly explain your reasoning in every problem.

1. Prove the principle of "casting out nines" for multiplication:
Let $n_1, n_2, \ldots, n_k$ be positive integers, and let $n = \Pi_{i=1}^{k} n_i$. For $i = 1, 2, \ldots, k$, let s_i be the sum of the digits of n_i, and let s be the sum of the digits of n. Then $s \equiv \Pi_{i=1}^{k} s_i \pmod 9$.

2. Use the Chinese Remainder Theorem to solve the following simultaneous congruence: $x \equiv 13 \pmod{42}$, $x \equiv 49 \pmod{165}$.

3. Find the sixth roots of $a = -64i$.

4. (a) State the Division Algorithm for polynomials with coefficients in a field F.

(b) Use the Division Algorithm to prove that the polynomial ring $F[x]$ is a principal ideal domain.

5. Which of the following polynomials are irreducible in $\mathbb{Q}[x]$?

(a) $x^{10} - 12$

(b) $x^4 - 3x^3 + 2x^2 - 4x - 6$

(c) $x^4 + 4x^2 - 3x + 5$

6. How many ideals does the ring $\mathbb{Z}_{15}$ have? List all the ideals of this ring.

7. (a) Prove that the quotient ring $K = \mathbb{Z}_3[x]/\langle x^3 + x^2 + 2\rangle$ is a field, by using a theorem from the course.

(b) Compute the multiplicative inverse of $a = \overline{2x^2 + 1} \in K$.

8. Prove that $\sqrt[5]{2}$ is not a constructible number.