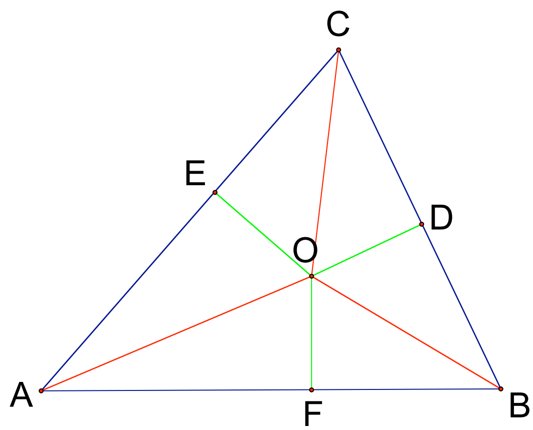
MATH 5200/7200 Exam 2 Solutions
10/28/08

1. Prove the following theorem. Write your proof in complete sentences, and give a reason for each step in your proof.

<u>Theorem</u>: If a triangle has the same circumcenter and incenter, then it is an equilateral triangle.



Proof: Let O be the circumcenter of the triangle ABC. Thus O is the center of the circumcircle c, and A, B, C lie on c. Therefore O is the same distance from the points A, B, and C. In other words, OA = OB = OC.

Suppose O is also the incenter of triangle ABC. Thus O is the center of the incircle d, and d is tangent to the sides of the triangle ABC. Let D on BC, E on AC, and F on AB be the points where the sides are tangent to the circle d. Then the segments OD, OE, and OF are radii of the incircle. Therefore these three segments have the same length, OD = OE = OF, and, by the definition of tangents, these three segments are perpendicular to the three sides of the triangle.

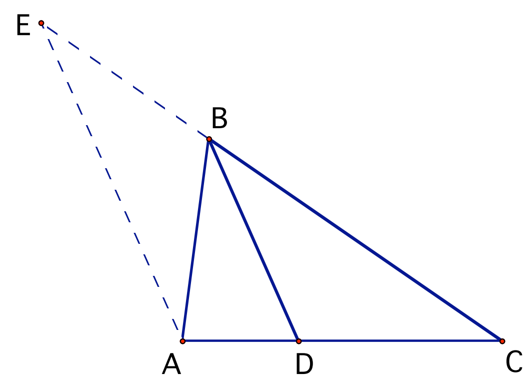
Therefore the six triangles AOF, BOF, BOD, COD, COE, AOE are right triangles. These right triangles have equal hypotenuses, OA = OB = OC, and one equal leg, OD = OE = OF. So by the Pythagorean Theorem the remaining legs are equal: AF = BF = BD = CD = CE = AE.

Now AB = AF + BF, BC = BD + CD, and AC = AE + CE, so we conclude that AB = BC = AC, and therefore ABC is an equilateral triangle.

2. Give reasons for each of the numbered steps in the following proof outline. The reasons for each step can involve one or more axioms or theorems, together with one or more previous steps in the proof outline.

Theorem: The bisector of an angle of a triangle divides the opposite side into segments that are proportional to the adjacent sides.

Proof. We show that if ABC is a triangle, then the bisector of the angle ABC intersects the side AC at a point D such that AD/DC = AB/BC.



1. Angle ABD equals angle DBC. Definition of angle bisector.
2. There is a line L through A parallel to the line BD. Existence of Parallel Theorem.
– Let E be the intersection of the line BC and the line L.
3. Angle ABD equals angle BAE. Step 2 and the Alternate Interior Angles Theorem.
4. Angle AEB equals angle DBC. Step 2 and the Corresponding Angles Theorem.
5. Angle AEB equals angle BAE. Steps 1, 3, 4.
6. EB = AB. Step 5 and the Isosceles Triangle Theorem.
7. Angle AEB equals angle AEC. The Angle Axiom.
8. Triangle AEC is similar to triangle DBC. Steps 7, 4, and the Angle-Angle Similarity Theorem.
9. AC/DC = EC/BC. Step 8 and definition of similar triangles.
– AC = AD + DC and EC = EB + BC, by the segment addition theorem.
10. AD/DC = EB/BC. Step 9, the previous (unnumbered) step, and algebra.
11. AD/DC = AB/BC. Steps 10 and 6.