MATH 5200/7200 Exam 1A Solutions

1. (a) What is the geometric mean of two positive real numbers x and y?

The geometric mean of x and y is $m = \sqrt{(xy)}$. In other words, m is the positive real number such that $m^2 = xy$.

(b) Given a segment AB and a point P on AB, explain how to construct a segment whose length is the geometric mean of AP and BP, using only the GSP Construct menu. You do not have to explain why your construction works.

First construct the midpoint M of the segment AB. Then construct a circle C with center M through the point A. Next construct the line L through the point P such that L is perpendicular to the segment AB. Let Q be one of the two intersections of the circle C and the line L. Then the length of the segment PQ is the geometric mean of AP and BP.

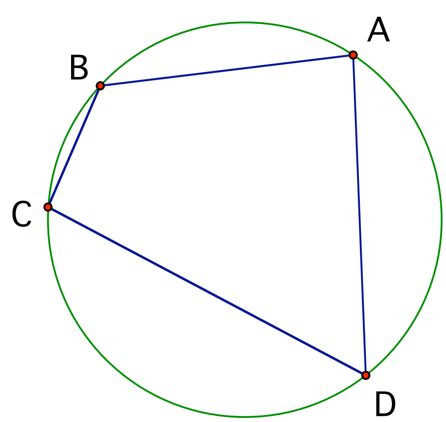
2. (a) State the arc angle theorem (inscribed angle theorem). (Don't prove it.)

If an angle is inscribed in a circle, then its measure is one-half the measure of the subtended arc.

In other words, if the points A, O, B lie on the circle c, then the measure of the angle AOB is 1/2 the arc angle measure of the arc of c with endpoints A and B that does not contain the point O.

(b) Use the arc angle theorem to prove that if a quadrilateral is inscribed in a circle, then the opposite angles of the quadrilateral are supplementary.

Let ABCD be a quadrilateral inscribed in a circle. We prove that the sum of angle A and angle C is 180 degrees. (The same argument applies to angles B and D.)

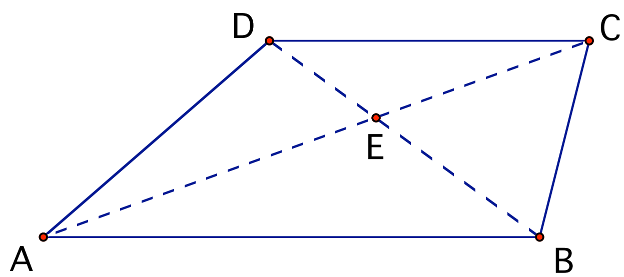


Angle A is angle DAB. By the arc angle theorem, angle DAB is 1/2 the arc angle of the arc BCD. Angle C is angle BCD. By the arc angle theorem, angle BCD is 1/2 the arc angle of the arc DAB. So

angle A + angle B = 1/2 [(arc angle BCD) + (arc angle DAB)].

Now the arcs BCD and DAB have endpoints B and D in common, and the union of these two arcs is the whole circle. Thus (arc angle BCD) + (arc angle (DAB) = 360 degrees. Therefore angle A + angle B = 180 degrees.

3. Prove that the diagonals of a trapezoid cut off proportional segments on each other. That is, for a trapezoid ABCD as in the figure below, if E is the intersection of the diagonals, then AE/EC = BE/ED. (The sides AB and DC are parallel.)



The line AC is a transversal of the parallel lines AB and DC. Therefore the alternate interior angles BAE and DCE are equal. Also, the angles BEA and DEC are equal, since they are opposite angles (so-called “vertical angles”) formed by the lines AC and BD. Therefore, by the angle-angle similarity theorem, the triangles ABE and CDE are similar. By the definition of similarity, the ratio of the corresponding sides AE and CE equals the ratio of the corresponding sides BE and DE. Thus AE/CE = BE/DE, or AE/EC = BE/ED.