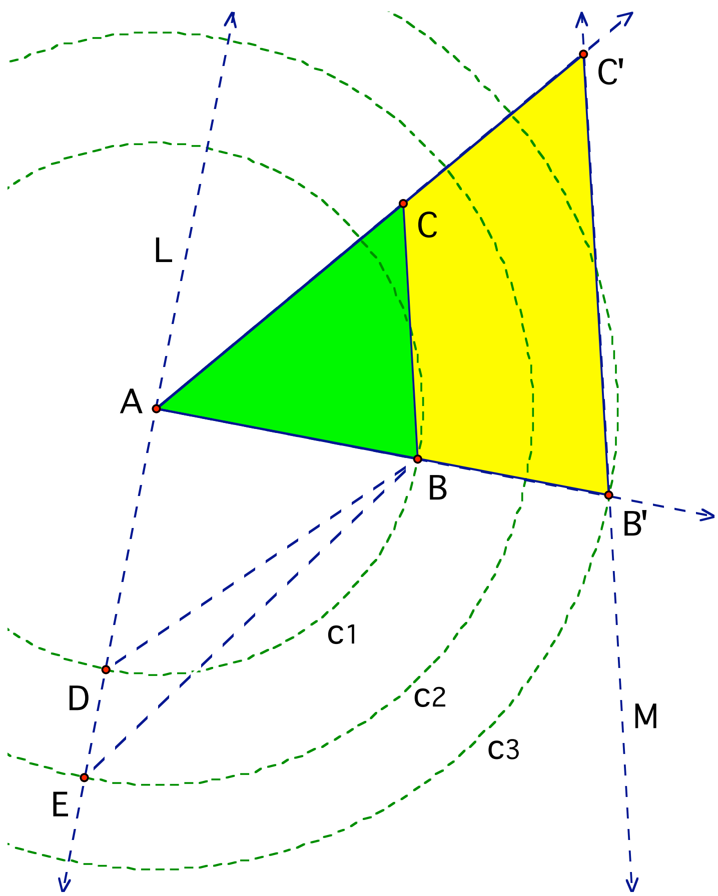
MATH 5200/7200 Exam 1B Solutions

In each of these problems, first do a Geometer's Sketchpad construction (using only the GSP Construct menu), and then prove your construction works.

1. Given a triangle ABC, construct a triangle A'B'C' so that A'B'C' is similar to ABC and the area of A'B'C' is three times the area of ABC.



Description of the construction. Let ABC be a triangle. Let A' = A. Let L be a line through A perpendicular to AB.

Let c_1 be a circle with center A that goes through point B. Let D be an intersection of L and c_1. Let c_2 be a circle with center A and radius DB. Let E be an intersection of L and c_2. Let c_3 be a circle with center A and radius EB.

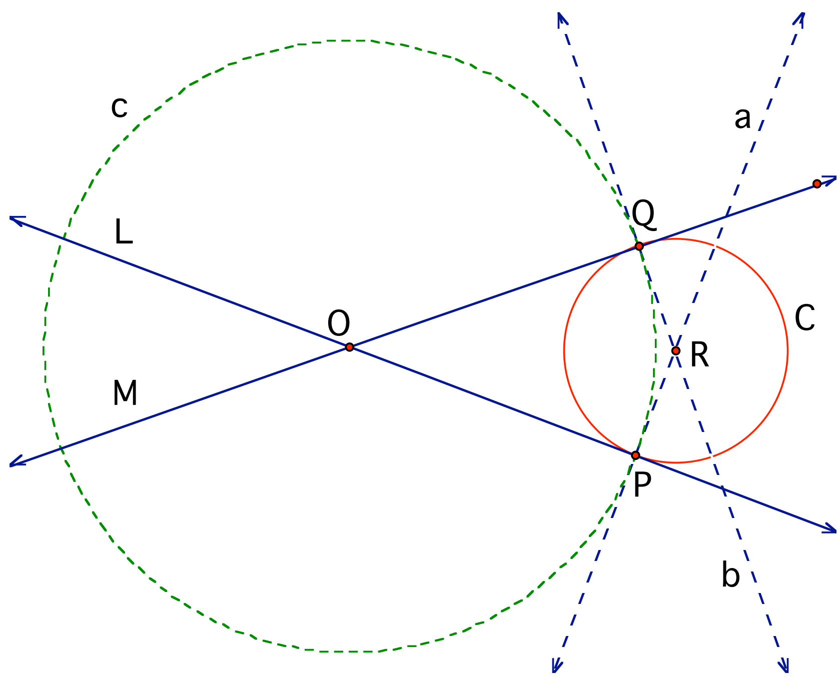
Let B' be the intersection of c_3 and the ray AB. Let M be the line through B' parallel to BC. Let C' be the intersection of M and the ray AC.

Then the area of triangle AB'C' is three times the area of triangle ABC.

<u>Proof that the construction works</u>. Let s be the length of segment AB. By construction (using the circle c_1), triangle ABD is an isosceles right triangle, with legs of length s. By the Pythagorean Theorem, the hypotenuse DB has length $\sqrt{2}$ s. Again by construction (using circle c_2), triangle ABE is a right triangle with legs of length s and $\sqrt{2}$ s, so by the Pythagorean Theorem, the hypotenuse EB has length $\sqrt{3}$ s. Thus AB' has length $\sqrt{3}$ s.

Now triangles ABC and AB'C' share the angle A, and angles B and B' are equal, since they are corresponding angles of the transversal AB of the parallel lines BC and B'C'. Thus by the angle-angle similarity theorem, the triangles ABC and AB'C' are similar. The similarity ratio is AB'/AB = $\sqrt{3}$ (from the construction of B' using circle c_3). So the area ratio is Area(AB'C')/Area(ABC) = $(\sqrt{3})^2$ = 3.

2. Given two lines L and M and a point P on the line L, construct a circle C so that P lies on C, and L and M are tangent to C.



<u>Description of the construction</u>. Case 1: L and M are not parallel. Let O be the intersection of L and M. Let c be the circle with center O that goes through the point P, and let Q be an intersection of c and the line M. Let a be the line through L perpendicular to P and let b be the line through Q perpendicular to M. Let R be the intersection of a and b. Let C be the circle with center R that goes through the point P. Then C is tangent to L at P and C is tangent to M at Q.

Case 2: L and M are parallel (and L is not equal to M). Let a be the line through P perpendicular to L. Let Q be the intersection of a and M. let R be the midpoint of the segment PQ. Let C be the circle with center R that goes through the point P. Then C is tangent to L at P and C is tangent to M at Q.

Case 3: L = M. Let a be the line through P perpendicular to L, and let R be any point on a other than P. Let C be the circle with center R that goes through the point P. Then C is tangent to L = M at P.

Proof that the construction works. Case 1: L and M are not parallel. By construction (using the circle c), OP = OQ. Consider the two triangles OPR and OQR. They are right triangles, with right angles at P and Q, respectively, by construction of lines a and b. They share side OR, and sides OP and OQ are equal. By the Pythagorean Theorem, sides RP and RQ are equal. Therefore Q lies on the circle with center R through the point P. Since OP is perpendicular to RP, the line L = OP is tangent to C at P. Since OQ is perpendicular to RQ, the line M = OQ is tangent to C at Q.

Case 2: L and M are parallel (and L is not equal to M). By construction of line a, RP is perpendicular to L, so C is tangent to L at P. Since a is perpendicular to L, and M is parallel to L, a is also perpendicular to M. So RQ is also perpendicular to M. Since R is the midpoint of PQ, RP = RQ, so Q lies on the circle C, and C is tangent to M at Q.

Case 3: L = M. By construction of the line a, RP is perpendicular to L, so circle C is tangent to L = M at P.

Note: If L and M are not parallel, and P is not equal to O (the intersection of L and M), then there are two circles through P that are tangent to L and M, one circle on either side of line L. In the construction, the circle c intersects the line M in two points, Q and Q'. Either of these points can be used to complete the construction.

If P = O, then there is no circle through P tangent to L and M.

Conclusion: Depending on the positions of L, M, and P, the number of circles C with the required properties can be 0, 1, 2, or infinite. These are the only possibilities.