MATH 5200/7200 Notes
Dr. McCrory

Dr. McCrory began class by saying that we will investigate spherical geometry this week. He said that he likes to think of Euclidean Geometry as the mother, and hyperbolic and spherical geometry as two sisters. They can both be understood by their comparisons.

Dr. McCrory said unfortunately, spherical geometry is not taught much any more (even though it is considered "***the*** Non-Euclidean Geometry"). Spherical geometry is much older than hyperbolic, and sometimes called Elliptic Geometry. It was first discovered by astrometry, ancient Greeks and Babylonians. However, we do not have much time for the history of spherical geometry in only a week, so let us begin.

Dr. McCrory says that Geometry of the Sphere is a GSP software for spherical geometry. However, it flattens out the spherical geometry, and makes it difficult to understand. So, we will not use the GSP program.

We will start with points. A point in Spherical Geometry is any point on the sphere. We may use a sphere of any radius, but conventionally we will use a radius of 1.

In Spherical Geometry, lines and (straight is an odd concept for Spherical Geometry), the shortest path on a sphere are called *geodesics.* Dr. McCrory showed an example of a geodesic on the globe from a flight from Atlanta to Tokyo. Flight goes directly over Anchorage, Alaska.

Dr. McCrory "So, geometrically, how do we describe these geodesics? How would you do a
3-D construction?"
Consider the concept of "lines" in Spherical Geometry

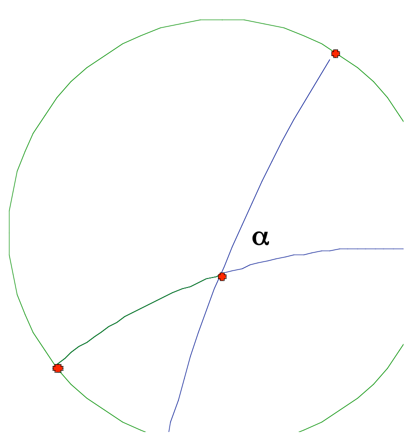
- Dr. McCrory said think about the center of the sphere.
- Dr. McCrory make a suggestion to go back to the example, airplanes take routes that correspond to the "*Great Circle*" routes, the Equator.
- There are an infinite number of great circles.
- *Antipodes*: the exact opposite point from a given point.
- The great circle has the same center as the sphere.
- Think of planes of great circles with a plane through the center of sphere.
- Geodesics are arcs of these great circles.
- Unlike Euclidean Geometry, there may be more than one shortest distance (especially if points are antipodes).
- Geodesics play the role of lines.
- A line is not infinite, it is a circle.
    - o There are no parallel lines, for any two great circles intersect at two points (as long as they are not the same line).
    - o So, we don't have the parallel axiom.

Next topic, Triangles.

- One difference is that you can make a triangle with three right angles.
    - Try to visualize taking an apple and cutting it into eighths, then you have eight congruent sections of an apple, the face, the skin of the apple is a triangle, and all three angles are 90 degrees.
    - Dr. McCrory makes a 90-90-90 equilateral (or isosceles) triangle on the globe. All three sides are a quarter of the circumference of the sphere.
- Mary Catherine asks are all equilateral triangles 90-90-90.
- Dr. McCrory, "No." It depends on how big the triangle is. Visualization shows an approximation where the angles look like 60, but it is a little bit larger. The smaller the geodesic triangle the angles approach 60 degrees and mirror Euclidean Geometry. The 90-90-90 is not the largest triangle you can make!

There are theorems that are true in Euclidean that are true in Spherical and many that are not.

- The most interesting one is the theorem about area of triangles. The sum of the angles in a triangle will always be greater than 180 degrees.
    - Allison asks is there an upper limit? Think about there is not just one arc between two points.
    - Dr. McCrory "I don't know if there is a maximum you can get. Well, there is a theoretical maximum."
    - So, what should be the relation between the sum of the angles and the area in Spherical? Here it is the efficiency, or angle excess. The sum of the angles of the triangle minus 180. (Remember Hyperbolic is 180 minus sum of the angles, and it is called the deficiency.)
        - Before we start the proof, everything will be in radians. The area of a geodesic triangle is the sum of the angles minus π .
        - $\textit{Area of geodesic triangle} = (\textit{Sum of the Angles}) - \pi$
        - To do this proof, understand the intersection of two great circles. The region in-between two great circles is called a lune. (A polygon with two sides) (Kind of like a slice of an orange) A lune has only two sides and two angles and the angles are equal.
    - Proof



What is the formula for area of a lune with an angle α ?

Think of it as a proportion, length vs. area

$$\frac{\alpha}{2\pi} = \frac{Area\,of\ Lune}{4\pi}$$

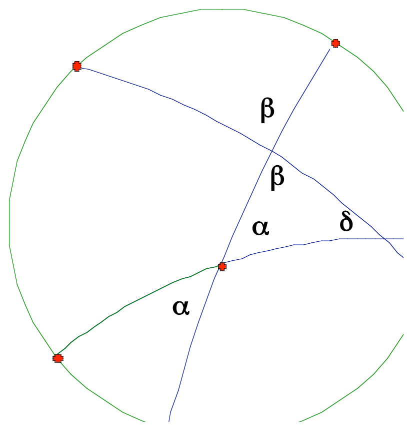
This is the basis for the proof about the area of a geodesic triangle.

We know, $2\alpha = Area\ of\ lune$

Now suppose there is a geodesic triangle on the sphere.

There should be three great circles, and every pair of great circles intersects at 3 points.

(Note: Another triangle just like this one is on the back of the circle)



Make the following lunes:

2 α lunes

2 β lunes

2 δ lunes

What happens if we add up all areas of the lunes?

All 6 area include the area of the triangle 6 times

$$2(Area\,\alpha\,lune) + 2(Area\,\beta\,lune) + 2(Area\,\delta\,lune) = Area\,of\ sphere + 4(Area\,of\ Triangle)$$

$$4\alpha + 4\beta + 4\delta = 4\pi + 4\,Area$$

$$\alpha + \beta + \delta = \pi + A$$

$$\alpha + \beta + \delta - \pi = Area$$

- Homework for Friday is not a GSP homework. Go through our axioms and basic theorems and see which hold in spherical geometry. It will be quite analogous in to hyperbolic geometry. Helpful to look at a tangible sphere with string to discover properties.