MATH 4010/6010
Background on field extensions
April 15, 2009

(1) If $f(x)$ is an irreducible polynomial in $F[x]$, then the quotient ring $F[x]/\langle f(x)\rangle$ is a field.

This is half of Theorem 4.2.3 on page 129. (The other half is the converse.)

(2) If If $f(x)$ is an irreducible polynomial in $F[x]$, K is an extension field of F, and α is a root of $f(x)$ in K, then the ring $F[\alpha]$ is isomorphic to $F[x]/\langle f(x)\rangle$, and hence $F[\alpha]$ is a field.

This is closely related to Theorem 4.2.4. The proof is as follows. Consider the evaluation homomorphism $\varphi : F[x]- > F[\alpha]$, $\varphi(p(x)) = p(\alpha)$. The homomorphism φ is surjective by definition of $F[\alpha]$. The kernel of φ is the set of $p(x)$ such that $p(\alpha) = 0$, and this ideal is $\langle f(x)\rangle$ by Corollary 4.1.3. So (2) follows from Theorem 4.2.2.

(3) If K is an extension field of F with $[K : F] = n$, and $\alpha \in K\backslash F$, then there is a polynomial $g(x)$ of degree n in $F[x]$ such that α is a root of $g(x)$. Hence there is an irreducible polynomial $f(x) \in F[x]$ such that α is a root of $f(x)$.

See Exercise 5.1.19.