MATH 4250/6250 Exam 1 Retest Solutions
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1. Compute the tangent, normal, and binormal vectors of the curve

$$\vec{\alpha}(t) = (\cos 2t, \sin 2t, t^2)$$

at $t = \pi$.

$\vec{\alpha}'(t) = (-2\sin 2t, 2\cos 2t, 2t),\ \vec{\alpha}'(\pi) = (0, 2, 2\pi)$
$\vec{\alpha}''(t) = (-4\cos 2t, -4\sin 2t, 2),\ \vec{\alpha}''(\pi) = (-4, 0, 2)$

$\vec{T} = \vec{\alpha}'/|\vec{\alpha}'| = (0, 2, 2\pi)/\sqrt{4+4\pi^2} = \frac{1}{\sqrt{1+\pi^2}}(0, 1, \pi)$

$\vec{B} = \vec{\alpha}' \times \vec{\alpha}''/|\vec{\alpha}' \times \vec{\alpha}''| = (4, -8\pi, 8)/4\sqrt{1+4\pi^2+4} = \frac{1}{\sqrt{5+4\pi^2}}(1, -2\pi, 2)$

$\vec{N} = \vec{B} \times \vec{T} = \frac{1}{\sqrt{(5+4\pi^2)(1+\pi^2)}}(-2\pi^2 - 2, -\pi, 1)$

2. (a) State the Frenet equations for a space curve.

Let $\vec{\alpha}$ be a unit speed curve with tangent $\vec{T}$, normal $\vec{N}$, binormal $\vec{B}$, curvature κ, and torsion τ. Then

$\vec{T}' = \kappa\vec{N}$
$\vec{N}' = -\kappa\vec{T} + \tau\vec{B}$
$\vec{B}' = -\tau\vec{N}$

(b) Prove that if $\vec{\alpha}$ is a regular space curve with constant torsion $\tau = 0$, then the image of $\vec{\alpha}$ lies in a plane. (Prove that if the binormal vector is constant, then $\vec{\alpha}$ lies in a plane.)

We assume that $\vec{\alpha}(s)$ is parametrized by arclength. Since $\tau = 0$, we have $\vec{B}' = -\tau\vec{N} = 0$, so $\vec{B}$ is constant.

Consider the function $f(s) = (\vec{\alpha}(s) - \vec{\alpha}(0)) \cdot \vec{B}$. We have

$$f'(s) = (\vec{\alpha}(s) - \vec{\alpha}(0))' \cdot \vec{B} + (\vec{\alpha}(s) - \vec{\alpha}(0)) \cdot \vec{B}' = \vec{\alpha}'(s) \cdot \vec{B} = \vec{T} \cdot \vec{B} = 0.$$

Thus $f(s)$ is constant. But $f(0) = (\vec{\alpha}(0) - \vec{\alpha}(0)) \cdot \vec{B} = 0$, so $f(s) = 0$ for all s. Thus $(\vec{\alpha}(s) - \vec{\alpha}(0)) \cdot \vec{B} = 0$ for all s, and $\vec{\alpha}(s)$ lies in the plane through the point $\vec{\alpha}(0)$ perpendicular to the vector $\vec{B}$.

(c) Prove that if $\vec{\alpha}$ is a regular space curve with constant torsion $\tau = 0$ and constant curvature $\kappa > 0$, then the image of $\vec{\alpha}$ lies in a circle of radius $1/\kappa$. (What is the center of the circle?)

Let $\vec{c}(s) = \vec{\alpha}(s) + \frac{1}{\kappa}\vec{N}(s)$. Since $\alpha(s)$ lies in the plane P through $\alpha(0)$ perpendicular to the constant vector $\vec{B}$, and $\vec{N}$ is perpendicular to $\vec{B}$, the point $\vec{c}(s)$ also lies in the plane P. Now

$$\vec{c}'(s) = \vec{\alpha}'(s) + \frac{1}{\kappa}\vec{N}'(s) = \vec{T} + \frac{1}{\kappa}(-\kappa\vec{T} + \tau\vec{B}) = \vec{T} - \frac{1}{\kappa}(\kappa\vec{T}) = 0,$$

so $\vec{c}(s) = \vec{c}$ is constant. Now $\vec{\alpha}(s) - \vec{c} = -\frac{1}{\kappa}\vec{N}(s)$, so $|\vec{\alpha}(s) - \vec{c}| = \frac{1}{\kappa}$, and hence $\alpha(s)$ lies on the circle in the plane P with center $\vec{c}$ and radius $\frac{1}{\kappa}$.

3. (a) Show that the function

$$\vec{x}(u,v) = (u, uv, e^v)$$

is differentiable, and that the partial derivatives $\vec{x}_u$ and $\vec{x}_v$ are independent. (You do not have to check that the domain of $\vec{x}$ is open or that $\vec{x}$ is injective.)

$\vec{x}_u = (1, v, 0)$ and $\vec{x}_v = (0, u, e^v)$, so $\vec{x}(u,v)$ is differentiable. The cross product $\vec{x}_u \times \vec{x}_v = (ve^v, -e^v, u)$ is never 0, since $e^v > 0$ for all v. Thus $\vec{x}_u$ and $\vec{x}_v$ are independent.

(b) Find an equation for the tangent plane to the surface $\vec{x}$ at the point $\vec{x}(2,1)$

(i) Parametric equation:

$$\vec{p} = \vec{x}(2,1) + s\vec{x}_u(2,1) + t\vec{x}_v(2,1),$$
$$(x,y,z) = (2,2,e) + s(1,1,0) + t(0,2,e).$$

(ii) Implicit equation:

$$(\vec{p} - \vec{x}(2,1)) \cdot (\vec{x}_u(2,1) \times \vec{x}_v(2,1)) = 0,$$
$$((x,y,z) - (2,2,e)) \cdot ((1,1,0) \times (0,2,e)) = 0,$$
$$(x-2, y-2, z-e) \cdot (e, -e, 2) = 0,$$
$$e(x-2) - e(y-2) + 2(z-e) = 0.$$

4. Compute the coordinate transformation $\varphi(s,t) = (u,v)$ for the coordinate patches

$$\vec{x}(s,t) = (s/t, st, s), \quad t > 0,$$
$$\vec{y}(u,v) = (u, v, \sqrt{uv}), \quad u > 0, \quad v > 0,$$

of the surface $S = \{(x,y,z) \mid xy = z^2\}$. Find the domain of φ, and verify that the derivative matrix $D\varphi$ is nonsingular. (You do not have to check that φ is bijective.)

[*Correction:* The coordinate patch $\vec{x}(s,t)$ has domain $t > 0$, $s > 0$, and the surface S is smooth only for $(x,y,z) \neq (0,0,0)$. This does not affect the solution of the problem.]

To compute the coordinate transformation $\varphi(s,t) = (u,v)$, we set $\vec{x}(s,t) = \vec{y}(u,v)$:

$$(s/t, st, s) = (u, v, \sqrt{uv}),$$
$$s/t = u, \quad st = v, \quad s = \sqrt{uv}.$$

Thus $\varphi(s,t) = (s/t, st)$, and the domain of φ is $t > 0$ and $s > 0$ (since $s = \sqrt{uv}$). We have

$$D\varphi = \begin{pmatrix} 1/t & -s/t^2 \\ t & s \end{pmatrix}$$

and

$$\det D\varphi = s/t + st/t^2 = 2s/t \neq 0,$$

since $s \neq 0$ and $t \neq 0$. Thus $D\varphi$ is nonsingular.