

1. Define the following invariants of surfaces:

(a) shape operator: Let M be a surface and let $\vec{p} \in M$. The *shape operator* $S_{\vec{p}}$ of M at \vec{p} is the linear transformation $S_{\vec{p}} : T_{\vec{p}}M \rightarrow T_{\vec{p}}M$ defined by $S_{\vec{p}}(\vec{v}) = -\nabla_{\vec{v}}\vec{U}$, where \vec{U} is the unit normal vector of M .

(b) principal curvatures: The *principal curvatures* of M at \vec{p} are the eigenvalues of the shape operator $S_{\vec{p}}$.

(c) Gaussian curvature: The *Gaussian curvature* K of M at \vec{p} is the determinant of the shape operator $S_{\vec{p}}$.

(d) mean curvature: The *mean curvature* H of M at \vec{p} is one-half the trace of the shape operator $S_{\vec{p}}$.

2. Prove that if a surface M contains a line L through the point \vec{p} on M , then $K(\vec{p}) \leq 0$, where $K(\vec{p})$ is the Gaussian curvature of M at \vec{p} . Fill in the details of the following proof outline:

(a) Let \vec{u} be a unit tangent vector to M at \vec{p} in the direction of the line L . Then, if $S_{\vec{p}}$ is the shape operator of M at \vec{p} , we have $S_{\vec{p}}(\vec{u}) \cdot \vec{u} = 0$.

Let $\vec{\alpha}(t) = \vec{p} + t\vec{u}$. Then $\vec{\alpha}(0) = \vec{p}$ and $\vec{\alpha}'(0) = \vec{u}$. By definition of the shape operator, $S_{\vec{p}}(\vec{u}) = \frac{d}{dt}(\vec{U}(\vec{\alpha}(t)))|_{t=0} = \vec{U}'(0)$. Now $\vec{U} \cdot \vec{u} = 0$ implies $\vec{U}' \cdot \vec{u} + \vec{U} \cdot \vec{u}' = 0$. Since \vec{u} is constant, we have $\vec{u}' = 0$, so we conclude that $\vec{U}' \cdot \vec{u} = 0$, or $S_{\vec{p}}(\vec{u}) \cdot \vec{u} = 0$.

(b) It follows that the normal curvature $\mathbf{k}(\vec{u}) = 0$.

We have $\mathbf{k}(\vec{u}) = S_{\vec{p}}(\vec{u}) \cdot \vec{u} = 0$.

(c) Euler's formula implies that the principal curvatures k_1 or k_2 of M at p have opposite signs or one of them is zero.

Euler's formula is $\mathbf{k}(\vec{u}) = k_1 \cos^2 \theta + k_2 \sin^2 \theta$, where k_1 and k_2 are the principal curvatures of M at \vec{p} ($k_1 = k_2$ if \vec{p} is an umbilic point), and θ is the angle from \vec{u}_1 to \vec{u} , where \vec{u}_1 is the eigenvector belonging to the eigenvalue k_1 . (If \vec{p} is an umbilic point then u_1 and u_2 is an arbitrary orthonormal basis for $T_{\vec{p}}M$.) Since $\mathbf{k}(\vec{u}) = 0$, we have $k_1 \cos^2 \theta = -k_2 \sin^2 \theta$. Now $\cos^2 \theta \geq 0$ and $\sin^2 \theta \geq 0$, and for all θ , $\cos \theta$ and $\sin \theta$ are not both zero. Therefore either $k_1 = 0$ or $k_2 = 0$ or k_1 and k_2 have opposite signs. (In particular, $k_1 = k_2$ if and only if they are both zero, and so \vec{p} is an umbilic point.)

(d) Therefore $K(p) \leq 0$.

We have $K = k_1 k_2$. If $k_1 = 0$ then $K = 0$, if $k_2 = 0$ then $K = 0$, and if k_1 and k_2 have opposite signs then $K < 0$.

3. (a) Compute the Gaussian curvature of the surface

$$\vec{x}(u, v) = (u, v, u^2/2 + v^3/3).$$

$$\vec{x}_u = (1, 0, u), \quad \vec{x}_v = (0, 1, v^2)$$

$$E = \vec{x}_u \cdot \vec{x}_u = 1 + u^2, \quad F = \vec{x}_u \cdot \vec{x}_v = uv^2, \quad G = \vec{x}_v \cdot \vec{x}_v = 1 + v^4$$

$$\vec{x}_u \times \vec{x}_v = (-u, -v^2, 1), \quad |\vec{x}_u \times \vec{x}_v| = \sqrt{u^2 + v^4 + 1}, \quad \vec{U} = (-u, -v^2, 1)/\sqrt{u^2 + v^4 + 1}$$

$$\vec{x}_{uu} = (0, 0, 1), \quad \vec{x}_{uv} = (0, 0, 0), \quad \vec{x}_{vv} = (0, 0, 2v)$$

$$l = \vec{U} \cdot \vec{x}_{uu} = 1/\sqrt{u^2 + v^4 + 1}, \quad m = \vec{U} \cdot \vec{x}_{uv} = 0, \quad n = \vec{U} \cdot \vec{x}_{vv} = 2v/\sqrt{u^2 + v^4 + 1}$$

$$K = \frac{ln - m^2}{EG - F^2} = \frac{2v/(u^2 + v^4 + 1)}{u^2 + v^4 + 1} = \frac{2v}{(u^2 + v^4 + 1)^2}$$

(b) For which (u, v) is $K = 0$, $K > 0$, $K < 0$?

$$K = 0 \iff v = 0, \quad K > 0 \iff v > 0, \quad K < 0 \iff v < 0.$$

4. Compute the principal curvatures for every point of the torus

$$\vec{x}(u, v) = ((2 + \cos u) \cos v, (2 + \cos u) \sin v, \sin u).$$

Shortcut: Assume the fact that the meridians (u -parameter curves) and parallels (v -parameter curves) are lines of curvature.

$$\vec{x}_u = (-\sin u \cos v, -\sin u \sin v, \cos u), \quad \vec{x}_v = (-(2 + \cos u) \sin v, (2 + \cos u) \cos v, 0)$$

$$\begin{aligned} \vec{x}_u \times \vec{x}_v &= (-(2 + \cos u) \cos u \cos v, -(2 + \cos u) \cos u \sin v, \\ &\quad -\sin u \cos^2 v(2 + \cos u) - \sin u \sin^2 v(2 + \cos u)) \\ &= (2 + \cos u)(-\cos u \cos v, -\cos u \sin v, -\sin u) \end{aligned}$$

$$\begin{aligned} |\vec{x}_u \times \vec{x}_v|^2 &= (2 + \cos u)^2(\cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \sin^2 u) = (2 + \cos u)^2 \\ |\vec{x}_u \times \vec{x}_v| &= (2 + \cos u) \end{aligned}$$

$$\vec{U} = \vec{x}_u \times \vec{x}_v / |\vec{x}_u \times \vec{x}_v| = (-\cos u \cos v, -\cos u \sin v, -\sin u)$$

$$S(\vec{x}_u) = -\frac{\partial \vec{U}}{\partial u} = -(\sin u \cos v, \sin u \sin v, -\cos u) = k_1 \vec{x}_u \implies k_1 = 1$$

$$S(\vec{x}_v) = -\frac{\partial \vec{U}}{\partial v} = -(\cos u \sin v, -\cos u \cos v, 0) = k_2 \vec{x}_v \implies k_2 = \frac{\cos u}{2 + \cos u}$$