MATH 4250/6250 Final Exam
Wednesday, May 6, 12:00 – 3:00 pm
C. McCrory

1. Compute the tangent vector, normal vector, binormal vector, curvature, and torsion of the curve

$$\vec{\alpha}(t) = (\cos t, \sin t, \cos 2t)$$

at $t = 0$.

2. Suppose the binormal vector of a unit speed space curve $\vec{\alpha}(s)$ makes a constant angle with a fixed vector $\vec{u}$, and the curvature κ and torsion τ are never zero. Prove that the ratio $\kappa(s)/\tau(s)$ is constant, by filling in the details of the following proof outline:

(i) Show $\vec{B} \cdot \vec{u} = a$, where a is a constant.
(ii) Show $\vec{N} \cdot \vec{u} = 0$.
(iii) Show $\vec{u} = a\vec{B} + f(s)\vec{T}$.
(iv) Take the derivative of (iii) and conclude that κ/τ is constant.

3. Find the equation of the tangent plane of the surface

$$\vec{x}(u, v) = (u, v, u^2 - v^3)$$

at the point $\vec{x}(1, -1)$.

4. Define the following invariants of surfaces:

(a) shape operator
(b) principal curvatures
(c) Gaussian curvature
(d) mean curvature

5. Compute the principal curvatures and the Gaussian curvature of the surface

$$\vec{x}(u, v) = (u^2, e^u \cos v, e^u \sin v), \quad -\infty < u < \infty, \quad 0 < v < 2\pi.$$

6. Compute the shape operator S of the helicoid

$$\vec{x}(u, v) = (u \cos v, u \sin v, v), \quad 0 < u < \infty, \quad 0 < v < 2\pi$$

at the point $\vec{x}(1, \pi)$. (Find the matrix of S with respect to the basis $\vec{x}_u$, $\vec{x}_v$.)

7. Let $\vec{x}(u, v)$ be a coordinate patch with $E = 1$ and $F = 0$. Prove that the u-parameter curves $\vec{\alpha}(u) = \vec{x}(u, v_0)$ (v_0 constant) are geodesics. (Hint: Compute the partial derivatives of E and F.)

8. (a) Give the following definitions:

(i) parallel vector field along a curve $\vec{\alpha}$ on a surface M

(ii) holonomy of a closed curve $\vec{\alpha}$ on a surface M

(b) State the angle excess theorem for a geodesic triangle.