

Geodesics locally minimize distance - sketch of a proof.

(1) If $\vec{\alpha}$ is a unit speed geodesic segment on a surface M , there is a coordinate chart \vec{x} on M such that $E = 1$, $F = 0$, the image of \vec{x} contains an initial segment of $\vec{\alpha}$, and this initial segment is a u -parameter curve in \vec{x} . (This is the hardest part of the proof. It involves setting up “geodesic coordinates,” a construction due to Gauss.)

(2) Conversely, if α is a u -parameter curve in a coordinate chart \vec{x} with $E = 1$ and $F = 0$, then $\vec{\alpha}$ is a unit speed geodesic. (The proof of this fact is just like the proof that a meridian on a surface of revolution is a geodesic.)

(3) If α is a u -parameter curve in a coordinate chart \vec{x} with $E = 1$ and $F = 0$, and $\vec{\beta}$ is another curve in this chart with the same initial and terminal points as $\vec{\alpha}$, then the length of $\vec{\beta}$ is greater than or equal to the length of $\vec{\alpha}$. (The proof is very similar to the proof that a straight line segment in the plane is the shortest curve between its endpoints.)