Parallel Transport
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Let $\vec{x}(u,v)$ is a coordinate chart for the surface M with $F=0$. Let $\vec{E}_1 = \vec{x}_u/|\vec{x}_u|$ and $\vec{E}_2 = \vec{x}_v/|\vec{x}_v|$. For every point $\vec{p}$ in the image of $\vec{x}$, the pair $(\vec{E}_1, \vec{E}_2)$ is an orthonormal basis for the tangent space of M at $\vec{p}$. Let $\vec{\alpha}(t)$ be a unit speed curve in the image of $\vec{x}$. Since $\vec{E}_1 \cdot \vec{E}_1 = 1$, $\vec{E}_2 \cdot \vec{E}_2 = 1$, and $\vec{E}_1 \cdot \vec{E}_2 = 0$, it follows that there is a scalar function $\omega(t)$ such that for all t, $\nabla_{\vec{\alpha}'(t)}\vec{E}_1 = \omega(t)\vec{E}_2$ and $\nabla_{\vec{\alpha}'(t)}\vec{E}_2 = -\omega(t)\vec{E}_1$.

Theorem. Let $\vec{\alpha}$ be a unit speed curve on the surface M, $\vec{\alpha} : [0,b] \to M$. Given a vector $\vec{V}_0 \in T_{\vec{\alpha}(0)}M$ there exists a unique parallel vector field $\vec{V}$ on M along $\vec{\alpha}$ such that $\vec{V}(0) = \vec{V}_0$.

Proof. If $\vec{V}$ is a parallel vector field along $\vec{\alpha}$, we can assume $\vec{V}$ is a unit vector field, and so we can write $\vec{V} = \cos\theta\,\vec{E}_1 + \sin\theta\,\vec{E}_2$. Then

$$\begin{aligned}\nabla_{\vec{\alpha}'}\vec{V} &= -\sin\theta\,\frac{d\theta}{dt}\vec{E}_1 + (\cos\theta)\omega\vec{E}_2 + \cos\theta\,\frac{d\theta}{dt}\vec{E}_2 - (\sin\theta)\omega\vec{E}_1.\\ &= -\sin\theta\left(\frac{d\theta}{dt}+\omega\right)\vec{E}_1 + \cos\theta\left(\frac{d\theta}{dt}+\omega\right)\vec{E}^2.\end{aligned}$$

Thus

$$\nabla_{\vec{\alpha}'}\vec{V} = 0 \iff \frac{d\theta}{dt} = -\omega,$$

and so $\vec{V}$ is uniquely determined by the condition

$$\theta(t) = \theta(0) - \int_0^t \omega(s)ds.$$

(If the image of $\vec{\alpha}$ is not contained in a single coordinate chart, then one subdivides the domain of $\vec{\alpha}$ and applies the preceding argument to each subinterval of the domain.) $\square$