MATH 4250/6250
Problem Set 12 solutions

<u>Problem set 12</u> (4/17/09)
Oprea: 3.3.10, 3.4.2, 5.5.2

(1) (3.3.10) <u>The pseudosphere</u> (surface of revolution of a tractrix)

$$\vec{x}(u,v) = (g(u), h(u)\cos v, h(u)\sin v)$$

$$\vec{\alpha}(u) = (g(u), h(u)), \quad g(u) = u, \quad h'(u) = \frac{-h(u)}{\sqrt{c^2 - h(u)^2}}$$

<u>Show</u> (a) $k_\mu = h'/c$

(b) $k_\pi = -1/ch'$

(c) $K = -1/c^2$

or $h' = \frac{-h}{\sqrt{c^2-h^2}}$

$\Rightarrow h'' = \frac{-h'c^2}{(c^2-h^2)^{3/2}}$

(a) $k_\mu = \dfrac{g''h' - g'h''}{((g')^2 + (h')^2)^{3/2}}$ (p. 120)

$$= \frac{-h''}{(1+(h')^2)^{3/2}}$$

$$= \frac{h' c^2}{(1+(h')^2)^{3/2}(c^2-h^2)^{3/2}}$$

$$= \frac{\dfrac{-h}{(c^2-h^2)^{\frac{1}{2}}}\, c^2}{\left(1+\dfrac{h^2}{c^2-h^2}\right)^{3/2}(c^2-h^2)^{3/2}}$$

$$= \frac{-hc^2}{(c^2-h^2)^{\frac{1}{2}}(c^2-h^2+h^2)^{3/2}}$$

$$= \frac{-h}{(c^2-h^2)^{\frac{1}{2}}c}$$

$$= \frac{h'}{c} \checkmark$$

(b) $k_\pi = \dfrac{g'}{h((g')^2+(h')^2)^{\frac{1}{2}}}$

(p. 120)

$$= \frac{1}{h\left(1+\dfrac{h^2}{c^2-h^2}\right)^{\frac{1}{2}}}$$

$$= \frac{(c^2-h^2)^{\frac{1}{2}}}{h(c^2-h^2+h^2)^{\frac{1}{2}}}$$

$$= \frac{(c^2-h^2)^{\frac{1}{2}}}{hc}$$

$$= -\frac{1}{h'c} \checkmark$$

(c) $K = k_\mu k_\pi$

$$= \frac{h'}{c}\left(-\frac{1}{h'c}\right)$$

$$= -\frac{1}{c^2} \checkmark$$

② (3.4.2) (helicoid)

$\vec{x}(u,v) = (u\cos v, u\sin v, v)$

$\vec{x}_u = (\cos v, \sin v, 0)$

$\vec{x}_v = (-u\sin v, u\cos v, 1)$

$\vec{x}_{uu} = (0,0,0)$

$\vec{x}_{uv} = (-\sin v, \cos v, 0)$

$\vec{x}_{vv} = (-u\cos v, -u\sin v, 0)$

$E = \vec{x}_u \cdot \vec{x}_u = \cos^2 v + \sin^2 v = 1$

$F = \vec{x}_u \cdot \vec{x}_v = -u\cos v\sin v + u\cos v\sin v = 0$

$G = \vec{x}_v \cdot \vec{x}_v = u^2\sin^2 v + u^2\cos^2 v + 1 = 1+u^2$

$\vec{x}_u \times \vec{x}_v = (\sin v, -\cos v, u\cos^2 v + u\sin^2 v)$
$= (\sin v, -\cos v, u)$

$|\vec{x}_u \times \vec{x}_v|^2 = \sin^2 v + \cos^2 v + u^2 = 1 + u^2$

$|\vec{x}_u \times \vec{x}_v| = \sqrt{1+u^2}$

$\vec{U} = \dfrac{1}{\sqrt{1+u^2}}(\sin v, -\cos v, u)$

$l = \vec{U} \cdot \vec{x}_{uu} = 0$

$m = \vec{U} \cdot \vec{x}_{uv} = \dfrac{1}{\sqrt{1+u^2}}(-\sin^2 v - \cos^2 v) = -\dfrac{1}{\sqrt{1+u^2}}$

$n = \vec{U} \cdot \vec{x}_{vv} = \dfrac{1}{\sqrt{1+u^2}}(-u\cos v\sin v, +u\cos v\sin v) = 0$

$$K = \frac{ln - m^2}{EG - F^2} = \frac{0 \cdot 0 - \frac{1}{1+u^2}}{1+u^2} = \frac{-1}{(1+u^2)^2}$$

$\vec{y}(u,v) = (u\cos v, u\sin v, \ln u)$ $\quad$ $g(u) = \ln u$, $h(u) = u$

(surface of revolution)

$$K = \frac{g'(g''h' - h''g')}{h((g')^2 + (h')^2)^2} \quad (p. 119)$$

$$= \frac{\frac{1}{u}\left(-\frac{1}{u^2}\cdot 1 - 0\cdot\frac{1}{u}\right)}{u\left(\frac{1}{u^2}+1\right)^2} = \frac{-\frac{1}{u^3}}{\frac{u(1+u^2)^2}{u^4}} = \frac{-\frac{1}{u^3}}{\frac{(1+u^2}{u^3}}$$

$$= -\frac{1}{(1+u^2)^2} \checkmark$$

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(3) (5.5.2) $\vec{x} = (u\cos v, u\sin v, v)$, $\vec{y} = (r\cos s, r\sin s,$

Show they're not isometric.

$$\begin{tikzcd} M \arrow[r, "I"] & N \\ D \arrow[u, "\vec{x}"] \arrow[r, "\varphi"] & E \arrow[u, "\vec{y}"'] \end{tikzcd}$$

$$\left\{ \begin{array}{l} \vec{x}_u \cdot \vec{x}_u = (DI)\vec{x}_u \cdot (DI)\vec{x} \\ \vec{x}_u \cdot \vec{x}_v = (DI)\vec{x}_u \cdot (DI)\vec{x}_v \\ \vec{x}_v \cdot \vec{x}_v = (DI)\vec{x}_v \cdot (DI)\vec{x} \end{array} \right.$$

(a) $K = -\dfrac{1}{(1+u^2)^2}$ $\qquad$ $K = -\dfrac{1}{(1+r^2)^2}$

$$-\frac{1}{(1+u^2)^2} = -\frac{1}{(1+r^2)^2}$$

$$(1+u^2)^2 = (1+r^2)^2$$

$$1+u^2 = 1+r^2$$

$$u^2 = r^2$$

$$\boxed{r = \pm u}$$

have

$r = \pm u \Rightarrow r_u = \pm 1$

(b) $E_{\vec{x}} = E_{\vec{y}}\, r_u^2 + G_{\vec{y}}\, s_u^2 \quad , \quad G_{\vec{x}} = E_{\vec{y}}\, r_v^2 + G_{\vec{y}}\, s_v^2$

$E_{\vec{x}} = 1$
$F_{\vec{x}} = 0$
$G_{\vec{x}} = 1 + u^2$

$\vec{y}_r = (\cos s, \sin s, \frac{1}{r})$
$\vec{y}_s = (-r\sin s, r\cos s, 0)$

$E_{\vec{y}} = \vec{y}_r \cdot \vec{y}_r = \cos^2 s + \sin^2 s + \frac{1}{r^2}$
$= 1 + \frac{1}{r^2}$

$\vec{F}_y = \vec{y}_r \cdot \vec{y}_s = -r\cos s \sin s + r\cos s \sin s$
$= 0$

$G_{\vec{y}} = \vec{y}_s \cdot \vec{y}_s = r^2\sin^2 s + r^2\cos^2 s = r^2$

$E_{\vec{x}} = E_{\vec{y}}\, r_u^2 + G_{\vec{y}}\, s_u^2$

$1 = (1 + \frac{1}{r^2})(1) + r^2 s_u^2$

$1 = 1 + \frac{1}{r^2} + r^2 s_u^2$

$-\frac{1}{r^2} = r^2 s_u^2$

$-\frac{1}{r^4} = s_u^2$

$\circledast$

Impossible

$G_{\vec{x}} = E_{\vec{y}}\, r_v^2 + G_{\vec{y}}\, s_v^2$

$1 + u^2 = (1 + \frac{1}{r^2}) \cdot 0 + r^2 s_v^2$

$1 + u^2 = r^2 s_v^2$

$1 + r^2 = r^2 s_v^2$

$1 + \frac{1}{r^2} = s_v^2$

$\pm\sqrt{1 + \frac{1}{r^2}} = s_v$