MATH 4250/6250

Problem Set 13, due 4/24 Solutions

✓A. Show class + textbook descriptions of the tractrix are equivalent.

Class: $h = Ae^{aw}$, $g = \pm \int_0^w \sqrt{1-a^2A^2e^{2at}}\,dt + D$

$$\frac{dg}{dw} = \pm\sqrt{1-a^2A^2e^{2aw}}$$

$$\left(\frac{dg}{dw}\right)^2 = 1 - a^2A^2e^{2aw}, \quad \frac{dh}{dw} = aAe^{aw} \quad \left(\frac{dh}{dw}\right)^2 = a^2A^2e^{2aw}$$

$$\left(\frac{dg}{dw}\right)^2 + \left(\frac{dg}{dh}\right)^2 = 1$$

Text: $g = u$, $\dfrac{dh}{du} = \dfrac{-h}{\sqrt{1-h^2}}$

$$\frac{dh}{du} = \frac{dh}{dw}\frac{du}{dw} \qquad \left(\frac{dh}{du}\right)^2 = \left(\frac{dh}{dw}\right)^2 \Big/ \left(\frac{du}{dw}\right)^2$$

$$= \left(\frac{dh}{dw}\right)^2 \Big/ 1 - \left(\frac{dh}{dw}\right)^2$$

$$= \frac{a^2A^2e^{2aw}}{1-a^2A^2e^{2aw}}$$

$$= \frac{a^2h^2}{1-a^2h^2} = \frac{h^2}{\frac{1}{a^2} - h^2}$$

$$= \frac{h^2}{c^2-h^2} \qquad \boxed{c = \frac{1}{a}}$$

$$h = \pm\frac{h}{\sqrt{c^2-h^2}}$$

$$h = -\frac{h}{\sqrt{c^2-h^2}} \checkmark$$

B. Oprea : (5.1.1) (5.1.2) (5.1.3) (5.1.9) (5.1.10) (5.1.12) (5.1

(5.1.1) Show $\kappa_\alpha^2 = k(\vec{\alpha}')^2 + \kappa_g^2$

$\vec{\alpha}$ unit speed $\Rightarrow \vec{\alpha}'' = \kappa_\alpha \vec{N}$ (defn. of κ_α and $\vec{N}$)

$\kappa_\alpha N \perp T \Rightarrow \kappa_\alpha \vec{N} = a\,\vec{U} + b\,\vec{U}\times\vec{T}$

$\kappa_g \overset{\text{def}}{=} \vec{\alpha}'' \cdot \vec{U}\times\vec{T} \Rightarrow b = \kappa_g$

$k(\vec{\alpha}') \overset{\text{def}}{=} S(\vec{\alpha}')\cdot\vec{\alpha}' = \vec{U}\cdot\vec{\alpha}'' \Rightarrow a = k(\vec{\alpha}'')$

$\boxed{\kappa_\alpha \vec{N} = k(\vec{\alpha}'')\,\vec{U} + \kappa_g\,\vec{U}\times\vec{T}}$ dot each side w. ~~itse~~ itself

$\kappa_\alpha^2 = (k(\vec{\alpha}''))^2 + (\kappa_g)^2$ ✓

(5.1.2) $\vec{x}(u,v) = ((R+r\cos u)\cos v, (R+r\cos u)\sin v, r\sin u)$

$\vec{U} = -(\cos u\cos v, \cos u \sin v, \sin u)$

top curve : $u = \frac{\pi}{2}$, $\cos u = 0$, $\sin u = 1$

$\vec{\alpha}(t) = (R\cos t, R\sin t, r)$

$\vec{\alpha}'(t) = (-R\sin t, R\cos t, 0)$ $|\vec{\alpha}'(t)| = R$.

$\vec{\alpha}(s) = (R\cos\frac{s}{R}, R\sin\frac{s}{R}, r)$

$\vec{T} = (-\sin\frac{s}{R}, \cos\frac{s}{R}, 0)$

$\frac{d\vec{T}}{ds} = (-\frac{1}{R}\cos\frac{s}{R}, -\frac{1}{R}\sin\frac{s}{R}, 0) = \kappa\vec{N}$

$\kappa = \frac{1}{R}$, $\vec{N} = (-\cos\frac{s}{R}, -\sin\frac{s}{R}, 0)$

$$\vec{U} = (-\cos u \cos v, -\cos u \sin v, \sin u)$$
$$= (0, 0, 1)$$
$$\vec{T} = (-\sin t, \cos t, 0)$$

$$\vec{U} \times \vec{T} = (-\cos t, -\sin t, 0)$$

$$\kappa \vec{N} \cdot \vec{U} \times \vec{T} = \frac{1}{R}(-\cos t, \sin t, 0) \cdot (-\cos t, -\sin t, 0)$$

$$\kappa_g = \frac{1}{R}$$

$$\kappa \vec{N} \cdot \vec{U} = (-\cos t, -\sin t, 0) \cdot (0,0,1) = 0 \Rightarrow k(\vec{\alpha}') = 0$$

$$\kappa^2 = k(\vec{\alpha}')^2 + (\kappa_g)^2$$
$$\left(\frac{1}{R}\right)^2 = 0^2 + \left(\frac{1}{R}\right)^2 \quad \checkmark$$

(5.1.3) $\vec{\alpha}(s)$ unit speed curve on M, $\vec{\alpha}(0) = \vec{0}$.

$$\vec{\beta}(s) = \vec{\alpha}(s) + \rho(s)\vec{U}_0$$

$$\vec{\beta}(s) = \vec{\alpha}(s) - (\vec{\alpha}(s) \cdot \vec{U}_0)\vec{U}_0 \qquad \text{Show } \kappa_g(\vec{\alpha}) = \kappa(\vec{\beta})$$

$$\vec{\beta}'(s) = \vec{\alpha}'(s) - (\vec{\alpha}'(s) \cdot \vec{U}_0)\vec{U}_0 \longrightarrow \left(\vec{\beta}'(0) = \vec{\alpha}'(0) - (\vec{\alpha}'(0) \cdot \vec{U}_0)\vec{U}\right.$$
$$= \vec{T} - (\vec{T} \cdot \vec{U}_0)\vec{U}_0 \qquad = \vec{\alpha}'(0) = \vec{T}_0$$

$$\vec{\beta}''(s) = \kappa_\alpha \vec{N} - (\kappa_\alpha N \cdot \vec{U}_0)\vec{U}_0$$

$$\vec{\beta}''(0) = \kappa_\alpha \vec{N} - k(\vec{\alpha}')\vec{U}_0$$

$$\kappa(\vec{\beta}) = \frac{|\vec{\beta}' \times \vec{\beta}''|}{|\vec{\beta}'|^3} = |\vec{T}_0 \times (\kappa_\alpha \vec{N}_0 - k(\vec{\alpha}')\vec{U}_0)|$$

$= |\kappa_\alpha \vec{N}_0 - k(\vec{x}')\vec{U}_0|$ $\qquad$ $\kappa_\alpha \vec{N}_0 = k(\vec{x}')\vec{U}_0 + \kappa_g \vec{U}_0 \times \vec{T}_0$

$= |\kappa_g \vec{U}_0 \times \vec{T}_0| = \kappa_g$ ✓

(5.1.9) $\vec{\alpha}$ curve on S^2_R. $\vec{\alpha}$ geodesic $\Leftrightarrow$ $\vec{\alpha}$ great circle.

($\Leftarrow$) $\vec{\alpha}$ great circle $\Rightarrow \vec{\alpha} \subset S^2_R \cap P$, P plane through O
so $P \perp S^2_R$. Have proved this gives a geodesic.
$(\vec{U}$ in P, $\vec{U} \perp \vec{T}$; $\vec{N}$ in P, $\vec{N} \perp \vec{T} \Rightarrow \vec{N} = \pm \vec{U} \Rightarrow \kappa_g = 0)$

($\Rightarrow$) $\vec{\alpha}$ unit speed. $\vec{\alpha}$ geodesic $\Rightarrow \vec{\alpha}'' \parallel \vec{U} \Rightarrow \vec{N} = \pm \vec{U}$
$\vec{N} = \pm \frac{1}{R}\vec{\alpha}$.

$\vec{N}' = \pm \frac{1}{R}\vec{\alpha}' \Rightarrow \kappa \vec{T} + \tau \vec{B} = \pm \frac{1}{R}\vec{T} \Rightarrow \tau = 0, \kappa = \frac{1}{R}$

$\Rightarrow \vec{\alpha}$ plane circle of radius R

$\vec{B} \perp \vec{N} \Rightarrow \vec{B} \perp \vec{U} \Rightarrow \vec{B} \perp \vec{\alpha}$ $\qquad$ $|\vec{\alpha}| = \vec{U}$

$\Rightarrow P$ goes through $\vec{O} \Rightarrow \vec{\alpha} \subset S^2_R \cap P \Rightarrow \alpha$ great circle.

(5.1.10) $\vec{\alpha}$ geod. on M, $\vec{\alpha} \subset P \Rightarrow \vec{\alpha}$ line of curvature.

$\vec{\alpha} \subset P \Rightarrow \tau = 0 \Rightarrow \vec{B}$ constant, $\vec{B} \perp P$

$\vec{\alpha}$ geodesic $\Rightarrow \vec{N} = \pm \vec{U}$, so $\vec{U}' = \pm \vec{N}' = \pm \kappa \vec{T} \pm \tau \vec{B}$
$\vec{U}' = \pm \kappa \vec{T}$
$S(\vec{T}) = \pm \kappa \vec{T}$
$\vec{T}$ eigenvector of S
$\vec{\alpha}$ line of curvature.

(5.1.12) Find the geodesics on the cylinder
$M: x^2+y^2=R^2$
$\vec{X}(u,v) = (R\cos u, R\sin u, bv)$

$\vec{\alpha}(t) = (R\cos u(t), R\sin u(t), bv(t))$

$\vec{\alpha}' = \left(-R\sin u \frac{du}{dt}, R\cos u \frac{du}{dt}, b\frac{dv}{dt}\right)$

$\vec{\alpha}'' = \left(-R\cos u\left(\frac{du}{dt}\right)^2 - R\sin u\left(\frac{d^2u}{dt^2}\right), -R\sin u\left(\frac{du}{dt}\right)^2 + R\cos u\left(\frac{d^2u}{dt^2}\right), b\frac{d^2v}{dt^2}\right)$

$U = (\cos u, \sin u, 0)$

$$\vec{\alpha}'' = -R\left(\frac{du}{dt}\right)^2\vec{U} + \underbrace{\left(-R\sin u \frac{d^2u}{dt^2}, +R\cos u\frac{d^2u}{dt^2}, b\frac{d^2v}{dt^2}\right)}_{=0}$$

$\Rightarrow \frac{d^2u}{dt^2} = 0, \quad \frac{d^2v}{dt^2} = 0$

$\Rightarrow \frac{du}{dt} = a \quad \frac{dv}{dt} = c$

$\Rightarrow \frac{du}{dt} = at+b \quad \frac{dv}{dt} = ct+d.$

So $\vec{\alpha}(t) = \vec{X}(at+b, ct+d)$ ✓

(5.1.13) nonsense, since the proof is false.