MATH 4250/6250
Problem Set 14 Solutions.

Problem Set 14 (4/30/09)
A. Oprea: (6.1.5), (6.1.9), (6.2.1), (6.3.3)
B. Angle excess formula.

(6.1.5)
(1) Surfaces of revolution

$$\vec{X}(u,v) = (g(u),\ h(u)\cos v,\ h(u)\sin v)$$

$$\vec{X}_u = (g',\ h'\cos v,\ h'\sin v)$$
$$\vec{X}_v = (0,\ -h\sin v,\ h\cos v)$$

$$\vec{X}_u \times \vec{X}_v = (hh'\cos^2 v + hh'\sin^2 v,\ -hg'\cos v\ -hg'\sin v)$$
$$= (hh',\ -hg'\cos v,\ -hg'\sin v)$$

$$|\vec{X}_u \times \vec{X}_v| = \sqrt{h^2(h')^2 + h^2(g')^2\cos^2 v + h^2(g')^2\sin^2 v}$$

$$= h\sqrt{(h')^2 + (g')^2}$$

$$A = \int_0^{2\pi}\int_a^b h\sqrt{(h')^2+(g')^2}\,du\,dv = \int_a^b\int_0^{2\pi} h\sqrt{(h')^2+(g')^2}\,dv\,du$$

$$= \int_a^b 2\pi h\sqrt{(h')^2+(g')^2}\,du. \longrightarrow$$

$u = x, \quad g(u) = x, \quad h(u) = f(x) \Rightarrow$

$$A = \int_a^b 2\pi\, f(x) \sqrt{1 + (f'(x))^2}\, dx \quad \checkmark$$

<u>Graphs of functions</u> $z = f(x,y)$.

$\vec{x}(u,v) = (u, v, f(u,v))$

$\vec{x}_u = (1, 0, \frac{\partial f}{\partial u})$

$\vec{x}_v = (0, 1, \frac{\partial f}{\partial v})$

$\vec{x}_u \times \vec{x}_v = (-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1)$

$|\vec{x}_u \times \vec{x}_v| = \sqrt{(\frac{\partial f}{\partial u})^2 + (\frac{\partial f}{\partial v})^2 + 1}$

$$A = \int_a^b \int_c^d \sqrt{1 + f_u^2 + f_v^2}\, du\, dv$$

$u = x, \ v = y \Rightarrow \quad A = \int_a^b \int_c^d \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy \quad \checkmark$

(6.1.9) $\boxed{\text{Torus}}$: $\vec{x}(u,v) = ((R + r\cos u)\sin v, (R + r\cos u)\cos v, r\sin u)$

<u>Class 4/24</u> : $|\vec{x}_u \times \vec{x}_v| = r(R + r\cos u)$

(p. 119) : $K = \dfrac{g'(g''h' - h''g')}{h((g')^2 + (h')^2)^2}$, $\quad g(u) = r\sin u$, $h(u) = R + r\cos u$

$g' = r\cos u \qquad h' = -r\sin u \qquad \longrightarrow$

$g'' = -r\sin u \qquad h'' = -r\cos u$

$$K = \frac{r\cos u\left((-r\sin u)(-r\sin u) - (-r\cos u)(r\cos u)\right)}{(R + r\cos u)(r^2\cos^2 u + r^2\sin^2 u)^2}$$

$$= \frac{r^3\cos u}{r^4(R + r\cos u)} = \frac{\cos u}{r(R + r\cos u)}$$

$$\int_M K = \int_0^{2\pi}\int_0^{2\pi} K\,|\vec{x}_u \times \vec{x}_v|\,du\,dv = \int_0^{2\pi}\int_0^{2\pi} \cos u\,dv\,du$$

$$= 2\pi \int_0^{2\pi} \cos u\,du \qquad 2\pi(-\sin u)\Big|_0^{2\pi} = 0.$$

$\boxed{\text{catenoid}}$: $\vec{x}(u,v) = \left(c\cosh\frac{u}{c}\cos v,\ c\cosh\frac{u}{c}\sin v,\ u\right)$

$$\vec{x}_u = \left(\sinh\frac{u}{c}\cos v,\ \sinh\frac{u}{c}\sin v,\ 1\right)$$

$$\vec{x}_v = \left(-c\cosh\frac{u}{c}\sin v,\ c\cosh\frac{u}{c}\cos v,\ 0\right)$$

$$\vec{x}_u \times \vec{x}_v = \left(-c\cosh\frac{u}{c}\cos v,\ -c\cosh\frac{u}{c}\sin v,\ c\cosh\frac{u}{c}\sinh\frac{u}{c}\right)$$

$$|\vec{x}_u \times \vec{x}_v| = \sqrt{c^2\cosh^2\frac{u}{c}\cos^2 v + c^2\cosh^2\frac{u}{c}\sin^2 v + c^2\cosh^2\frac{u}{c}\sinh^2\frac{u}{c}}$$

$$= c\sqrt{\cosh^2\frac{u}{c}\left(1 + \sinh^2\frac{u}{c}\right)} = c\sqrt{\cosh^4\frac{u}{c}}$$

$$= c\cosh^2\frac{u}{c}.$$

$g = u \qquad h = c\cosh\frac{u}{c}$

$g' = 1 \qquad h' = \sinh\frac{u}{c}$

$g'' = 0 \qquad h'' = \frac{1}{c}\cosh\frac{u}{c}$

$$K = \frac{1\left(0 - \frac{1}{c}\cosh\frac{u}{c}\right)}{c\cosh\frac{u}{c}\left(1 + \sinh^2\frac{u}{c}\right)^2}$$

$$= \frac{-\frac{1}{c}\cosh\frac{u}{c}}{c\cosh^5\frac{u}{c}} = -\frac{1}{c^2\cosh^4\frac{u}{c}}$$

$$\int_M K = \int_0^{2\pi}\int_{-\infty}^{\infty} -\frac{1}{c^2\cosh^4\frac{u}{c}}\, c\cosh^2\frac{u}{c}\; du\, dv$$

$$= -\int_0^{2\pi}\int_{-\infty}^{\infty}\frac{1}{c\cosh^2\frac{u}{c}}\,du\,dv = \int_{-\infty}^{\infty}\int_0^{2\pi}\frac{1}{c\cosh^2\frac{u}{c}}\,dv\,du$$

$$= -2\pi\int_{-\infty}^{\infty}\frac{1}{c}\operatorname{sech}^2\frac{u}{c}\,du$$

$$= -2\pi\left(\tanh\frac{u}{c}\right)\Big|_{-\infty}^{\infty}$$

$$= -2\pi\lim_{a\to\infty}\left(\tanh\frac{a}{c} - \tanh\left(-\frac{a}{c}\right)\right)$$

$$= -2\pi\left(1-(-1)\right) = -4\pi.$$

(6.2.1) Show $\nabla^{\mathbb{R}^3}_{\vec{\alpha}'}\vec{\alpha}' = \vec{\alpha}''$.

$$\nabla^{\mathbb{R}^3}_{\vec{v}}\vec{Z} = \frac{d}{dt}\left(Z(\vec{\alpha}(t))\right)\Big|_{t=t_0}, \quad \vec{\alpha}(t_0) = \vec{p}, \quad \vec{\alpha}'(t_0) = \vec{v}$$

$\vec{v}$ vector based at $\vec{p}$

$$\nabla^{\mathbb{R}^3}_{\vec{\alpha}'}\vec{\alpha}' = \nabla^{\mathbb{R}^3}_{\vec{\alpha}'(t_0)}\vec{Z} \qquad \vec{Z}(\vec{\alpha}(t)) = \vec{\alpha}'(t)$$

$$= \frac{d}{dt}\left(\vec{Z}(\vec{\alpha}(t))\right)\Big|_{t=t_0}$$

$$= \frac{d}{dt}\left(\vec{\alpha}'(t)\right)\Big|_{t=t_0} = \vec{\alpha}''(t_0)$$

(6.3.3) (a) $\vec{\alpha}$ geodesic, $\vec{V}$ parallel along $\vec{\alpha}$
$\Rightarrow$ $\vec{V}$ makes constant angle with $\vec{\alpha}'$

(b) $\vec{V}$ parallel along $\vec{\alpha}$, $\vec{W}$ vector field along $\vec{\alpha}$ of constant length, $\vec{W}$ parallel along $\vec{\alpha}$ $\Leftrightarrow$ angle between $\vec{V}$ and $\vec{W}$ constant.

<u>Note</u> (b) ($\Rightarrow$) implies (a), since $\vec{\alpha}$ is a geodesic $\Leftrightarrow$ $\vec{\alpha}'$ is parallel along $\vec{\alpha}$.

(b) Let θ be the angle between $\vec{V}$ and $\vec{W}$.

$$\vec{V}\cdot\vec{W} = |\vec{V}|\,|\vec{W}|\cos\theta$$

θ constant $\Leftrightarrow$ $\vec{V}\cdot\vec{W}$ constant (since $|\vec{V}|, |\vec{W}|$ const)

$$\Leftrightarrow \nabla^{M}_{\vec{\alpha}'}(\vec{V}\cdot\vec{W}) = 0$$

ie $\nabla^{M}_{\vec{\alpha}'}(\vec{V}\cdot\vec{W}) = (\nabla^{M}_{\vec{\alpha}'}\vec{V})\cdot\vec{W} + \vec{V}\cdot(\nabla^{M}_{\vec{\alpha}'}\vec{W}) = 0$

Since $\vec{V}$ parallel, $\nabla^{M}_{\vec{\alpha}'}\vec{V} = 0$, so

we have θ constant $\Leftrightarrow$ $\vec{V}\cdot(\nabla^{M}_{\vec{\alpha}'}\vec{W}) = 0$

$\Leftrightarrow$ $\nabla^{M}_{\vec{\alpha}'}\vec{W} = 0$ (ie $\vec{W}$ parallel)

<u>or</u> $\nabla^{M}_{\vec{\alpha}'}\vec{W} \neq 0$ and $\nabla^{M}_{\vec{\alpha}'}\vec{W} \perp \vec{V}$

But $\vec{W}\cdot\vec{W}$ const. $\Rightarrow$ $\nabla^{M}_{\vec{\alpha}'}\vec{W} \perp \vec{W}$, so

$\nabla^{M}_{\vec{\alpha}'}\vec{W} \perp \vec{V} \Rightarrow \vec{W} = c\vec{V} \Rightarrow \vec{W}$ parallel.

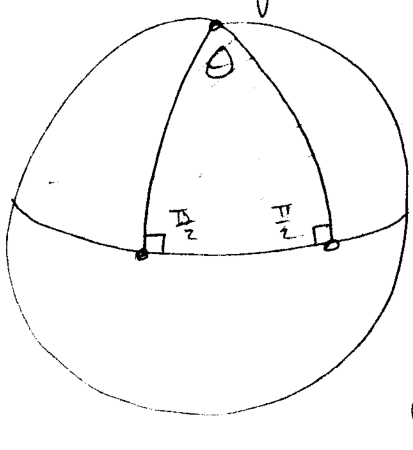
Use this to show that the holonomy around a curve does not depend on the parallel vector field along the curve.

Let $\alpha : [a,b] \to M$

Let $\vec{V}, \vec{W}$ be two parallel vector fields along α. Let θ be the constant angle from $\vec{V}$ to $\vec{W}$.

If the angle from $\vec{V}(a)$ to $\vec{W}(a)$ is θ and the angle from $\vec{V}(b)$ to $\vec{W}(b)$ is θ, then the angle from $\vec{V}(a)$ to $\vec{V}(b)$ equals the angle from $\vec{W}(a)$ to $\vec{W}(b)$.

B. Check the angle excess formula for a geodesic triangle on a sphere with one vertex at the north pole and two vertices on the equator.



The triangle has angles $\frac{\pi}{2}, \frac{\pi}{2}, \theta$ $\quad (0 < \theta < 2\pi)$

So the angle excess is

$$\left(\frac{\pi}{2} + \frac{\pi}{2} + \theta\right) - \pi = \theta.$$

We want to show $\int_\Delta K = \theta$.

$K = \frac{1}{R^2}$

$$\vec{x}(u,v) = (R\cos u \cos v, R\cos u \sin v, R\sin u) \qquad -\frac{\pi}{2} < u < \frac{\pi}{2}$$
$$0 < v < 2\pi$$

$$|\vec{x}_u \times \vec{x}_v| = R^2 \cos u$$

$$\int_\Delta K = \int_0^\theta \int_0^{\frac{\pi}{2}} \frac{1}{R^2} R^2 \cos u \, du\, dv = \int_0^{\frac{\pi}{2}} \int_0^\theta \cos u \, dv\, du$$

$$= \theta \int_0^{\frac{\pi}{2}} \cos u \, du = \theta \left(\sin u\right)_0^{\frac{\pi}{2}} = \theta \checkmark$$